4.

Resolution

Goal

Deductive reasoning in language as close as possible to full FOL

$\neg, \ \wedge, \ \vee, \ \exists, \forall$

Knowledge Level:

given KB, α, determine if KB $\models \alpha$.

or given an open $\alpha[x_1,x_2,...x_n]$, find $t_1,t_2,...t_n$ such that KB $\models \alpha[t_1,t_2,...t_n]$

When KB is finite $\{\alpha_1, \alpha_2, ..., \alpha_k\}$

KB $\models \alpha$

iff $\models [(\alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_k) \supset \alpha]$

iff KB $\cup \{\neg\alpha\}$ is unsatisfiable

iff KB $\cup \{\neg\alpha\}$ $\models$ FALSE

where FALSE is something like $\exists x.(x \neq x)$

So want a procedure to test for validity, or satisfiability, or
for entailing FALSE.

Will now consider such a procedure (first without quantifiers)

Clausal representation

Formula = set of clauses

Clause = set of literals

Literal = atomic sentence or its negation

positive literal and negative literal

Notation:

If ρ is a literal, then $\bar{\rho}$ is its complement

$\overline{p} \Rightarrow \neg p \qquad \overline{\neg p} \Rightarrow p$

To distinguish clauses from formulas:

[and] for clauses: $[p, \overline{r}, s]$ { and } for formulas: $\{ [p, \overline{r}, s], [p, r, s], [\overline{p}] \}$

[] is the empty clause {} is the empty formula

So {} is different from {[]}!

Interpretation:

Formula understood as <u>conjunction</u> of clauses

Clause understood as <u>disjunction</u> of literals

Literals understood normally

$\{[p,\neg q], [r], [s]\}$ represents $((p \vee \neg q) \wedge r \wedge s)$

[ ] represents FALSE

CNF and DNF

Every propositional wff α can be converted into a formula α' in <u>C</u>onjunctive <u>N</u>ormal <u>F</u>orm (CNF) in such a way that $\models \alpha \equiv \alpha'$.

1. eliminate $\supset$ and $\equiv$ using $(\alpha \supset \beta) \twoheadrightarrow (\neg\alpha \vee \beta)$ etc.
2. push $\neg$ inward using $\neg(\alpha \wedge \beta) \twoheadrightarrow (\neg\alpha \vee \neg\beta)$ etc.
3. distribute $\vee$ over $\wedge$ using $((\alpha \wedge \beta) \vee \gamma) \twoheadrightarrow ((\alpha \vee \gamma) \wedge (\beta \vee \gamma))$
4. collect terms using $(\alpha \vee \alpha) \twoheadrightarrow \alpha$ etc.

Result is a conjunction of disjunction of literals.

an analogous procedure produces DNF,
a disjunction of conjunction of literals

We can identify CNF wffs with clausal formulas

$(p \vee \neg q \vee r) \wedge (s \vee \neg r) \twoheadrightarrow \{ [p, \neg q, r], [s, \neg r] \}$

So: given a finite KB, to find out if KB $\models \alpha$, it will be sufficient to

1. put $(\text{KB} \wedge \neg\alpha)$ into CNF, as above
2. determine the satisfiability of the clauses

Resolution rule of inference

Given two clauses, infer a new clause:

From clause $\{ p \} \cup C_1$,
and $\{\neg p \} \cup C_2$,
infer clause $C_1 \cup C_2$.

$C_1 \cup C_2$ is called a <u>resolvent</u> of input clauses with respect to p.

Example:

clauses $[w, r, q]$ and $[w, s, \neg r]$ have $[w, q, s]$ as resolvent wrt r.

Special Case:

$[p]$ and $[\neg p]$ resolve to [] (the C_1 and C_2 are empty)

A <u>derivation</u> of a clause c from a set S of clauses is a sequence $c_1, c_2, ..., c_n$ of clauses, where $c_n = c$, and for each c_i, either

1. $c_i \in S$, or
2. c_i is a resolvent of two earlier clauses in the derivation

Write: $S \rightarrow c$ if there is a derivation

Rationale

Resolution is a symbol-level rule of inference, but has a connection to knowledge-level logical interpretations

Claim: Resolvent is <u>entailed</u> by input clauses.

Suppose $\Im \models (p \vee \alpha)$ and $\Im \models (\neg p \vee \beta)$

Case 1: $\Im \models p$

then $\Im \models \beta$, so $\Im \models (\alpha \vee \beta)$.

Case 2: $\Im \not\models p$

then $\Im \models \alpha$, so $\Im \models (\alpha \vee \beta)$.

Either way, $\Im \models (\alpha \vee \beta)$.

So: $\{(p \vee \alpha), (\neg p \vee \beta)\} \models (\alpha \vee \beta)$.

Special case:

$[p]$ and $[\neg p]$ resolve to [ ],

so $\{[p], [\neg p]\} \models$ FALSE

that is: $\{[p], [\neg p]\}$ is unsatisfiable

Derivations and entailment

Can extend the previous argument to derivations:

If $S \rightarrow c$ then $S \models c$

Proof: by induction on the length of the derivation.
Show (by looking at the two cases) that $S \models c_i$.

But the converse does not hold in general

Can have $S \models c$ without having $S \rightarrow c$.

Example: $\{[\neg p]\} \models [\neg p, \neg q]$ i.e. $\neg p \models (\neg p \vee \neg q)$
but no derivation

However.... Resolution *is* refutation complete!

Theorem: $S \rightarrow []$ iff $S \models []$

Result will carry over to quantified clauses (later)

sound and complete when restricted to [ ]

So for any set *S* of clauses: *S* is unsatisfiable iff $S \rightarrow []$.

Provides method for determining satisfiability: search all derivations for [].

So provides a method for determining all entailments

A procedure for entailment

To determine if KB $\models \alpha$,

- put KB, $\neg\alpha$ into CNF to get *S*, as before
- check if $S \rightarrow []$.

If KB = {}, then we are testing the validity of α

Non-deterministic procedure

1. Check if [] is in *S*.

   If yes, then return **UNSATISFIABLE**

2. Check if there are two clauses in *S* such that they resolve to produce a clause that is not already in *S*.

   If no, then return **SATISFIABLE**

3. Add the new clause to *S* and go to 1.

Note: need only convert KB to CNF once

- can handle multiple queries with same KB
- after addition of new fact α, can simply add new clauses α' to KB

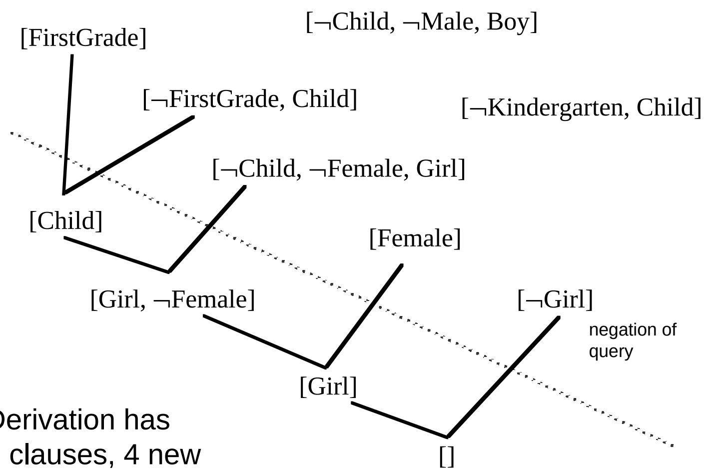
So: good idea to keep KB in CNF

Example 1

KB

FirstGrade
FirstGrade ⊃ Child
Child ∧ Male ⊃ Boy
Kindergarten ⊃ Child
Child ∧ Female ⊃ Girl
Female

Show that KB |= Girl



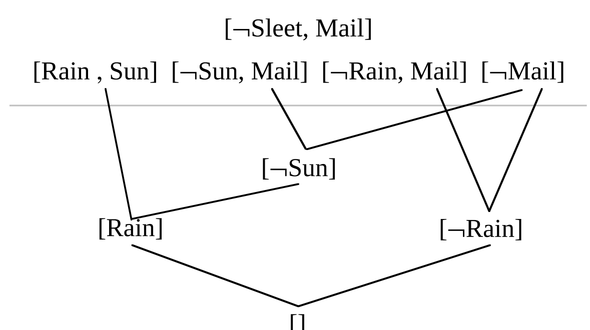
Derivation has
9 clauses, 4 new

Example 2

KB

(Rain ∨ Sun)
(Sun ⊃ Mail)
((Rain ∨ Sleet) ⊃ Mail)

Show KB |= Mail



Note: every clause
not in *S* has 2 parents

Similarly KB |≠ Rain

Can enumerate all resolvents given ¬Rain,
and [] will not be generated

Quantifiers

Clausal form as before, but atom is $P(t_1, t_2, ..., t_n)$, where t_i may contain variables

Interpretation as before, but variables are understood *universally*

Example: $\{ [P(x), \neg R(a,f(b,x))], [Q(x,y)] \}$

interpreted as

$\forall x \forall y \{[R(a,f(b,x)) \supset P(x)] \wedge Q(x,y)\}$

Substitutions: $\theta = \{v_1/t_1, v_2/t_2, ..., v_n/t_n\}$

Notation: If ρ is a literal and θ is a substitution, then $\rho\theta$ is the result of the substitution (and similarly, $c\theta$ where c is a clause)

Example: $\theta = \{x/a, y/g(x,b,z)\}$

$P(x,z,f(x,y))\,\theta = P(a,z,f(a,g(x,b,z)))$

A literal is <u>ground</u> if it contains no variables.

A literal ρ is an <u>instance</u> of ρ', if for some θ, $\rho = \rho'\theta$.

Generalizing CNF

Resolution will generalize to handling variables

Ignore = for now

But to convert wffs to CNF, we need three additional steps:

1. eliminate $\supset$ and $\equiv$
2. push $\neg$ inward using also $\neg\forall x.\alpha \Rightarrow \exists x.\neg\alpha$ etc.
3. standardize variables: each quantifier gets its own variable

   e.g. $\exists x[P(x)] \wedge Q(x) \Rightarrow \exists z[P(z)] \wedge Q(x)$ where z is a new variable
4. eliminate all existentials *(discussed later)*
5. move universals to the front using $(\forall x\alpha) \wedge \beta \Rightarrow \forall x(\alpha \wedge \beta)$

   where β does not use x
6. distribute $\vee$ over $\wedge$
7. collect terms

Get universally quantified conjunction of disjunction of literals

then drop all the quantifiers...

First-order resolution

Main idea: a literal (with variables) stands for all its instances; so allow all such inferences

So given $[P(x,a), \neg Q(x)]$ and $[\neg P(b,y), \neg R(b,f(y))]$,
want to infer $[\neg Q(b), \neg R(b,f(a))]$ among others

since $[P(x,a), \neg Q(x)]$ has $[P(b,a), \neg Q(b)]$ and
$[\neg P(b,y), \neg R(b,f(y))]$ has $[\neg P(b,a), \neg R(b,f(a))]$

Resolution:

Given clauses: $\{\rho_1\} \cup C_1$ and $\{\overline{\rho_2}\} \cup C_2$.

Rename variables, so that distinct in two clauses.

For any θ such that $\rho_1\theta = \rho_2\theta$, can infer $(C_1 \cup C_2)\theta$.

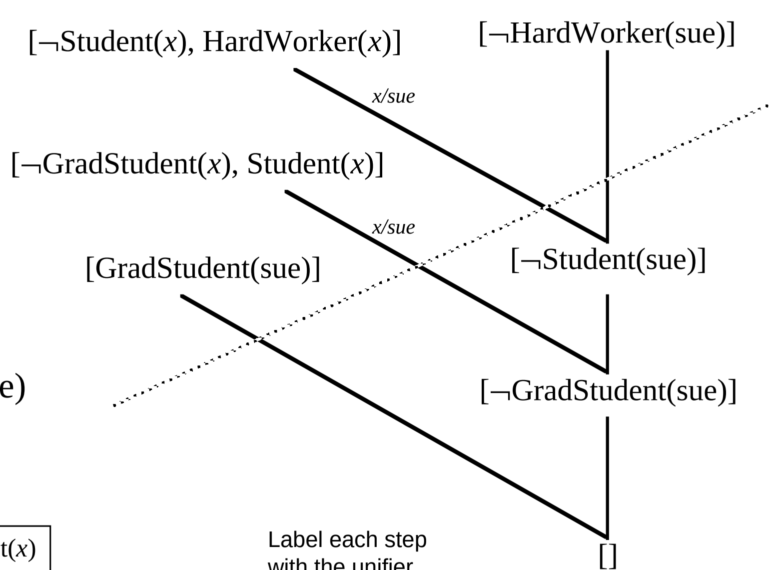
We say that ρ_1 <u>unifies</u> with ρ_2 and that θ is a <u>unifier</u> of the two literals

Resolution derivation: as before

Theorem: $S \rightarrow []$ iff $S \models []$ iff S is unsatisfiable

Note: There are pathological examples where a slightly more general definition of Resolution is required. We ignore them for now...

Example 3



$$\text{KB} \overset{?}{\models} \text{HardWorker(sue)}$$

KB

$\forall x$ GradStudent(x) $\supset$ Student(x)
$\forall x$ Student(x) $\supset$ HardWorker(x)
GradStudent(sue)

Label each step with the unifier

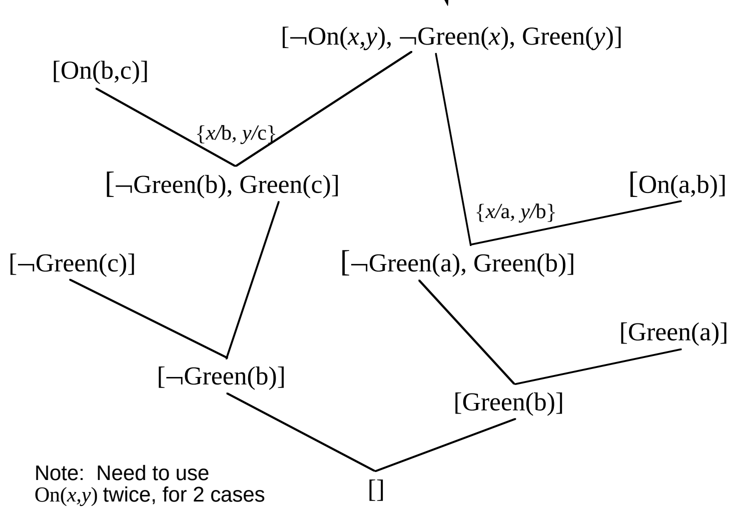
Point to relevant literals in clauses

The 3 block example

KB = {On(a,b), On(b,c), Green(a), ¬Green(c)}   already in CNF

Query = $\exists x \exists y[\mathrm{On}(x,y) \wedge \mathrm{Green}(x) \wedge \neg\mathrm{Green}(y)]$

Note: ¬Q has no existentials, so yields



Arithmetic

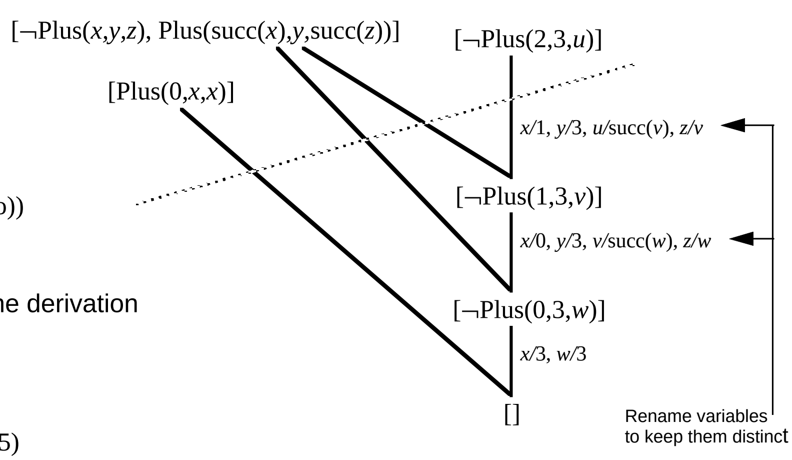
KB: $\mathrm{Plus}(\mathrm{zero},x,x)$

$\mathrm{Plus}(x,y,z) \supset \mathrm{Plus}(\mathrm{succ}(x),y,\mathrm{succ}(z))$

Q: $\exists u\ \mathrm{Plus}(2,3,u)$

For readability,
we use
- 0 for zero,
- 1 for succ(zero),
- 2 for succ(succ(zero))

etc.



Can find the answer in the derivation

$u/\mathrm{succ}(\mathrm{succ}(3))$

that is: $u/5$

Can also derive Plus(2,3,5)

Answer predicates

In full FOL, we have the possibility of deriving $\exists x P(x)$ without being able to derive $P(t)$ for any t.

e.g. the three-blocks problem

$\exists x \exists y[\mathrm{On}(x,y) \wedge \mathrm{Green}(x) \wedge \neg\mathrm{Green}(y)]$

but cannot derive which block is which

Solution: answer-extraction process

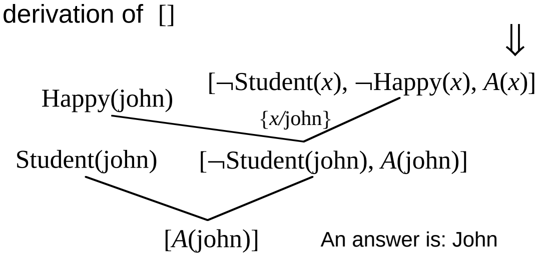
- replace query $\exists x P(x)$ by $\exists x[P(x) \wedge \neg A(x)]$

  where A is a new predicate symbol called the answer predicate

- instead of deriving [], derive any clause containing just the answer predicate
- can always convert to and from a derivation of []

KB: Student(john)
Student(jane)
Happy(john)

Q: $\exists x[\mathrm{Student}(x) \wedge \mathrm{Happy}(x)]$



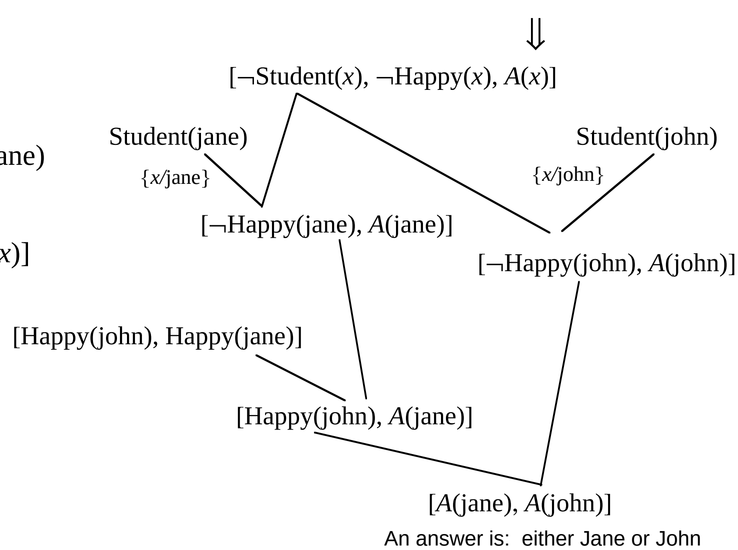
Disjunctive answers

KB:

Student(john)
Student(jane)
Happy(john) ∨ Happy(jane)

Query:

$\exists x[\mathrm{Student}(x) \wedge \mathrm{Happy}(x)]$



Note:

- can have variables in answer
- need to watch for Skolem symbols... (next)

Skolemization

So far, converting wff to CNF ignored existentials

e.g. $\exists x \forall y \exists z P(x,y,z)$

Idea: names for individuals claimed to exist, called Skolem constant and function symbols

there exists an x, call it a

for each y, there is a z, call it $f(y)$

get $\forall y P(a,y,f(y))$

So replace $\forall x_1(\ldots\forall x_2(\ldots\forall x_n(\ldots\exists y[\ldots\ y\ \ldots]\ \ldots)\ldots)\ldots)$

by $\forall x_1(\ldots\forall x_2(\ldots\forall x_n(\ \ldots\ \ [\ldots\ f(x_1,x_2,\ldots,x_n)\ \ldots]\ \ldots)\ldots)\ldots)$

f is a new function symbol that appears nowhere else

Skolemization does not preserve equivalence

e.g. $\not\models \exists x P(x) \equiv P(a)$

But it does preserve satisfiability

α is satisfiable iff α' is satisfiable (where α' is the result of Skolemization)

sufficient for resolution!

Variable dependence

Show that $\exists x \forall y R(x,y) \models \forall y \exists x R(x,y)$

show $\{\exists x \forall y R(x,y),\ \neg\forall y \exists x R(x,y)\}$ unsatisfiable

$\exists x \forall y R(x,y) \twoheadrightarrow \forall y R(a,y)$

$\neg\forall y \exists x R(x,y) \twoheadrightarrow \exists y \forall x \neg R(x,y) \twoheadrightarrow \forall x \neg R(x,b)$

then $\{\ [R(a,y)],\ [\neg R(x,b)]\ \} \rightarrow []$ with $\{x/a,\ y/b\}$.

Show that $\forall y \exists x R(x,y) \not\models \exists x \forall y R(x,y)$

show $\{\forall y \exists x R(x,y),\ \neg\exists x \forall y R(x,y)\}$ satisfiable

$\forall y \exists x R(x,y) \twoheadrightarrow \forall y R(f(y),y)$

$\neg\exists x \forall y R(x,y) \twoheadrightarrow \forall x \exists y \neg R(x,y) \twoheadrightarrow \forall x \neg R(x,\mathrm{g}(x))$

then get $\{\ [R(f(y),y)],\ [\neg R(x,g(x))]\ \}$

where the two literals do not unify

Note: important to get dependence of variables correct

$R(f(y),y)$ vs. $R(a,y)$ in the above

A problem

[LessThan(x,y), ¬LessThan(succ(x),y)]

[¬LessThan(0,0)]

x/0, y/0

[¬LessThan(1,0)]

x/1, y/0

[¬LessThan(2,0)]

x/2, y/0

...

KB:

LessThan(succ(x),y) ⊃ LessThan(x,y)

Query:

LessThan(zero,zero)

Should fail since KB |≠ Q

Infinite branch of resolvents

cannot use a simple depth-first procedure to search for []

Undecidability

Is there a way to detect when this happens?

No! FOL is very powerful

can be used as a full programming language

just as there is no way to detect in general when a program is looping

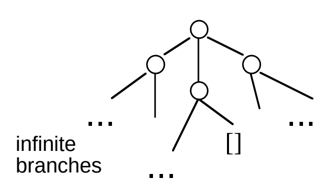
There can be no procedure that does this:

Proc[*Clauses*] =

If *Clauses* are unsatisfiable
then return YES
else return NO

However: Resolution is complete

some branch will contain [], for unsatisfiable clauses



So breadth-first search guaranteed to find []

search may not terminate on satisfiable clauses

Overly specific unifiers

In general, no way to guarantee efficiency, or even termination

later: put control into users' hands

One thing that can be done:

reduce redundancy in search, by keeping search as general as possible

Example

$\ldots, P(g(x),f(x),z)] \quad [\neg P(y,f(w),a), \ldots$

unified by

$\theta_1 = \{x/b, y/g(b), z/a, w/b\}$ gives $P(g(b),f(b),a)$

and by

$\theta_2 = \{x/f(z), y/g(f(z)), z/a, w/f(z)\}$ gives $P(g(f(z)),f(f(z)),a)$.

Might not be able to derive the empty clause from clauses having overly specific substitutions

wastes time in search!

Most general unifiers

θ is a <u>most general unifier</u> (MGU) of literals ρ_1 and ρ_2 iff

1. θ unifies ρ_1 and ρ_2
2. for any other unifier θ', there is a another substitution θ^* such that $\theta' = \theta\theta^*$

Note: composition $\theta\theta^*$ requires applying θ^* to terms in θ

for previous example, an MGU is

$\theta = \{x/w, y/g(w), z/a\}$

for which

$\theta_1 = \theta\{w/b\}$

$\theta_2 = \theta\{w/f(z)\}$

Theorem: Can limit search to most general unifiers only without loss of completeness (with certain caveats)

Computing MGUs

Computing an MGU, given a set of literals $\{\rho_i\}$

usually only have two literals

1. Start with $\theta := \{\}$.
2. If all the $\rho_i\theta$ are identical, then done; otherwise, get disagreement set, *DS*

   e.g $P(a,f(a,g(z),\ldots$ $P(a,f(a,u,\ldots$

   disagreement set, $DS = \{u, g(z)\}$
3. Find a variable $v \in DS$, and a term $t \in DS$ not containing v. If not, fail.
4. $\theta := \theta\{v/ t\}$
5. Go to 2

Note: there is a better *linear* algorithm

Herbrand Theorem

Some 1st-order cases can be handled by converting them to a propositional form

Given a set of clauses *S*

- the Herbrand universe of *S* is the set of all terms formed using only the function symbols in *S* (at least one)

  e.g., if *S* uses (unary) f, and c, d, $U = \{c, d, f(c), f(d), f(f(c)), f(f(d)), f(f(f(c))), \ldots\}$
- the Herbrand base of *S* is the set of all $c\theta$ such that $c \in S$ and θ replaces the variables in c by terms from the Herbrand universe

Theorem: *S* is satisfiable iff Herbrand base is

(applies to Horn clauses also)

Herbrand base has no variables, and so is essentially *propositional*, though usually infinite

- finite, when Herbrand universe is finite

  can use propositional methods (guaranteed to terminate)
- sometimes other “type” restrictions can be used to keep the Herbrand base finite

  include $f(t)$ only if t is the correct type

Resolution is difficult!

First-order resolution is not guaranteed to terminate.

What can be said about the propositional case?

- Shown by Haken in 1985 that there are unsatisfiable clauses $\{c_1, c_2, ..., c_n\}$ such that the *shortest* derivation of [] contains on the order of 2^n clauses
  - Even if we could always find a derivation immediately, the most clever search procedure will still require *exponential* time on some problems

Problem just with resolution?

- Probably not.
- Determining if a set of clauses is satisfiable was shown by Cook in 1972 to be NP-complete
  - No easier than an extremely large variety of computational tasks
  - Roughly: any search task where what is searched for can be verified in polynomial time can be recast as a satisfiability problem
    - » satisfiability
    - » does graph of cities allow for a full tour of size ≤ *k* miles?
    - » can N queens be put on an N×N chessboard all safely? and many, many more....
- Satisfiability is believed by most people to be unsolvable in polynomial time

SAT solvers

In the propositional case, procedures have been proposed for determining the satisfiability of a set of clauses that appear to work much better in practice than Resolution.

The most popular is called DP (or DPLL) based on ideas by Davis, Putnam, Loveland and Logemann. (See book for details.)

These procedures are called SAT solvers as they are mostly used to find a satisfying interpretation for clauses that are satisfiable.

related to constraint satisfaction programs (CSP)

Typically they have the property that if they *fail* to find a satisfying interpretation, a Resolution derivation of [ ] can be reconstructed from a trace of their execution.

so worst-case exponential behaviour, via Haken's theorem!

One interesting counter-example to this is the procedure GSAT, which has different limitations. (Again, see the book.)

Implications for KR

Problem: want to produce entailments of KB as needed for immediate action

full theorem-proving may be too difficult for KR!

need to consider other options ...

- giving control to user e.g. procedural representations (later)
- less expressive languages e.g. Horn clauses (and a major theme later)

In some applications, it is reasonable to wait

e.g. mathematical theorem proving, where we care about specific formulas

Best to hope for in general: reduce <u>redundancy</u>

main example: MGU, as before

but many other strategies (as we will see)

ATP: automated theorem proving

- area of AI that studies strategies for automatically proving difficult theorems
- main application: mathematics,but relevance also to KR

Strategies

1. Clause elimination

- pure clause

  contains literal ρ such that ρ does not appear in any other clause

  clause cannot lead to []

- tautology

  clause with a literal and its negation

  any path to [] can bypass tautology

- subsumed clause

  a clause such that one with a subset of its literals is already present

  path to [] need only pass through short clause

  can be generalized to allow substitutions

2. Ordering strategies

many possible ways to order search, but best and simplest is

- unit preference

  prefer to resolve unit clauses first

  Why? Given unit clause and another clause, resolvent is a smaller one ➟ []

Strategies 2

3. Set of support

   KB is usually satisfiable, so not very useful to resolve among clauses with only ancestors in KB

   contradiction arises from interaction with ¬Q

   always resolve with at least one clause that has an ancestor in ¬Q

   preserves completeness (sometimes)

4. Connection graph

   pre-compute all possible unifications

   build a graph with edges between any two unifiable literals of opposite polarity

   label edge with MGU

   Resolution procedure:

   repeatedly: select link
   compute resolvent
   inherit links from parents after substitution

   Resolution as search: find sequence of links L_1, L_2, ... producing []

Strategies 3

5. Special treatment for equality

   instead of using axioms for =

   relexitivity, symmetry, transitivity, substitution of equals for equals

   use new inference rule: <u>paramodulation</u>

   from $\{(t{=}s)\} \cup C_1$ and $\{P(\ldots\, t' \ldots)\} \cup C_2$

   where $t\theta = t'\theta$

   infer $\{P(\ldots\, s \ldots)\}\theta \;\cup\; C_1\theta \;\cup\; C_2\theta$.

   collapses many resolution steps into one

   see also: theory resolution (later)

6. Sorted logic

   terms get sorts:

   x: Male mother:[Person → Female]

   keep taxonomy of sorts

   only unify $P(s)$ with $P(t)$ when sorts are compatible

   assumes only “meaningful” paths will lead to []

Finally...

7. Directional connectives

given $[\neg p, q]$, can interpret as either

from p, infer q (forward)

to prove q, prove p (backward)

procedural reading of $\supset$

In 1st case: would only resolve $[\neg p, q]$ with $[p, ...]$ producing $[q, ...]$

In 2nd case: would only resolve $[\neg p, q]$ with $[\neg q, ...]$ producing $[\neg p, ...]$

Intended application:

forward: $\mathrm{Battleship}(x) \supset \mathrm{Gray}(x)$

do not want to try to prove something is gray
by trying to prove that it is a battleship

backward: $\mathrm{Person}(x) \supset \mathrm{Has}(x,\mathrm{spleen})$

do not want to conclude the spleen property for
each individual inferred to be a person

This is the starting point for the procedural representations (later)