5.

Reasoning with Horn Clauses

Horn clauses

Clauses are used two ways:

- as disjunctions: (rain ∨ sleet)
- as implications: (¬child ∨ ¬male ∨ boy)

Here focus on 2nd use

Horn clause = at most one +ve literal in clause

- positive / definite clause = exactly one +ve literal

  e.g. $[\neg p_1, \neg p_2, ..., \neg p_n, q]$

- negative clause = no +ve literals

  e.g. $[\neg p_1, \neg p_2, ..., \neg p_n]$ and also [ ]

Note: $[\neg p_1, \neg p_2, ..., \neg p_n, q]$ is a representation for
$(\neg p_1 \vee \neg p_2 \vee ... \vee \neg p_n \vee q)$ or $[(p_1 \wedge p_2 \wedge ... \wedge p_n) \supset q]$

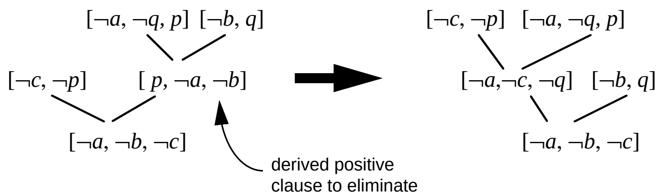
so can read as: If p_1 and p_2 and ... and p_n then q

and write as: $p_1 \wedge p_2 \wedge ... \wedge p_n \Rightarrow q$ or $q \Leftarrow p_1 \wedge p_2 \wedge ... \wedge p_n$

Resolution with Horn clauses

Only two possibilities:

It is possible to rearrange derivations of negative clauses so that all new derived clauses are negative



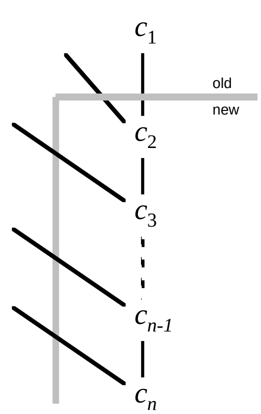
Further restricting resolution

Can also change derivations such that each derived clause is a resolvent of the previous derived one (negative) and some positive clause in the original set of clauses

- Since each derived clause is negative, one parent must be positive (and so from original set) and one parent must be negative.
- Chain backwards from the final negative clause until both parents are from the original set of clauses
- Eliminate all other clauses not on this direct path

This is a recurring pattern in derivations

- See previously:
  - example 1, example 3, arithmetic example
- But not:
  - example 2, the 3 block example



SLD Resolution

An <u>SLD-derivation</u> of a clause c from a set of clauses S is a sequence of clause $c_1, c_2, \ldots c_n$ such that $c_n = c$, and

1. $c_1 \in S$
2. c_{i+1} is a resolvent of c_i and a clause in S

Write: $S \overset{\text{SLD}}{\Rightarrow} c$

SLD means S(elected) literals
L(inear) form
D(efinite) clauses

Note: SLD derivation is just a special form of derivation and where we leave out the elements of S (except c_1)

In general, cannot restrict ourselves to just using SLD-Resolution

Proof: $S = \{[p, q], [p, \neg q], [\neg p, q]\ [\neg p, \neg q]\}$. Then $S \rightarrow []$.

Need to resolve some [ρ] and [$\overline{\rho}$] to get [].
But S does not contain any unit clauses.

So will need to derive both [ρ] and [$\overline{\rho}$] and then resolve them together.

Completeness of SLD

However, for Horn clauses, we can restrict ourselves to SLD-Resolution

Theorem: SLD-Resolution is refutation complete for Horn clauses: $H \rightarrow []$ iff $H \overset{\text{SLD}}{\Rightarrow} []$

So: H is unsatisfiable iff $H \overset{\text{SLD}}{\Rightarrow} []$

This will considerably simplify the search for derivations

Note: in Horn version of SLD-Resolution, each clause in the $c_1, c_2, \ldots, c_n$, will be negative

So clauses H must contain at least one negative clause, c_1 and this will be the only negative clause of H used.

Typical case:

- KB is a collection of positive Horn clauses
- Negation of query is the negative clause

Example 1 (again)

SLD derivation

$[\neg \text{Girl}]$

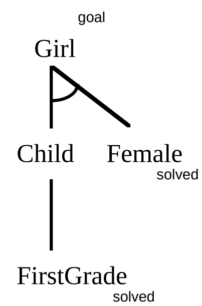
$[\neg \text{Child}, \neg \text{Female}]$

$[\neg \text{Child}]$

$[\neg \text{FirstGrade}]$

$[]$

alternate representation



KB

FirstGrade
FirstGrade $\supset$ Child
Child $\wedge$ Male $\supset$ Boy
Kindergarten $\supset$ Child
Child $\wedge$ Female $\supset$ Girl
Female

A goal tree whose nodes are atoms, whose root is the atom to prove, and whose leaves are in the KB

Show KB $\cup \{\neg \text{Girl}\}$ unsatisfiable

Prolog

Horn clauses form the basis of Prolog

$\text{Append}(\text{nil},y,y)$

$\text{Append}(x,y,z) \Rightarrow \text{Append}(\text{cons}(w,x),y,\text{cons}(w,z))$

What is the result of appending [c] to the list [a,b] ?

$\text{Append}(\text{cons}(a,\text{cons}(b,\text{nil})), \text{cons}(c,\text{nil}), u)$ goal

$u / \text{cons}(a,u')$

$\text{Append}(\text{cons}(b,\text{nil}), \text{cons}(c,\text{nil}), u')$

$u' / \text{cons}(b,u'')$

$\text{Append}(\text{nil}, \text{cons}(c,\text{nil}), u'')$

solved: $u'' /$ $\text{cons}(c,\text{nil})$

So goal succeeds with $u = \text{cons}(a,\text{cons}(b,\text{cons}(c,\text{nil})))$
that is: Append([a b],[c],[a b c])

With SLD derivation, can always extract answer from proof

$H \models \exists x\, \alpha(x)$

iff

for some term t, $H \models \alpha(t)$

Different answers can be found by finding other derivations

Back-chaining procedure

Solve[$q_1, q_2, ..., q_n$] = /* to establish conjunction of q_i */

If n=0 then return **YES**; /* empty clause detected */

For each $d \in$ KB do

If d = [$q_1, \neg p_1, \neg p_2, ..., \neg p_m$] /* match first q */

and /* replace q by -ve lits */

Solve[$p_1, p_2, ..., p_m, q_2, ..., q_n$] /* recursively */

then return **YES**

end for; /* can't find a clause to eliminate q */

Return **NO**

Depth-first, left-right, back-chaining

- depth-first because attempt p_i before trying q_i
- left-right because try q_i in order, 1,2, 3, ...
- back-chaining because search from goal q to facts in KB p

This is the execution strategy of Prolog

First-order case requires unification *etc.*

Problems with back-chaining

Can go into infinite loop

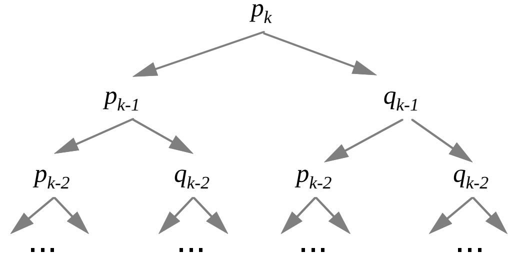
tautologous clause: [p , $\neg p$] (corresponds to Prolog program with p :- p).

Previous back-chaining algorithm is inefficient

Example: Consider 2n atoms, $p_0, ..., p_{n-1}, q_0, ..., q_{n-1}$ and $4n$-4 clauses

$(p_{i-1} \Rightarrow p_i)$, $(q_{i-1} \Rightarrow p_i)$, $(p_{i-1} \Rightarrow q_i)$, $(q_{i-1} \Rightarrow q_i)$.

With goal p_k the execution tree is like this



Solve[p_k] eventually fails after 2^k steps!

Is this problem inherent in Horn clauses?

Forward-chaining

Simple procedure to determine if Horn KB |= q.

main idea: mark atoms as solved

1. If q is marked as solved, then return **YES**
2. Is there a $\{p_1, \neg p_2, \ldots, \neg p_n\} \in$ KB such that $p_2, \ldots, p_n$ are marked as solved, but the positive lit p_1 is not marked as solved?
   - no: return **NO**
   - yes: mark p_1 as solved, and go to 1.

FirstGrade example:

Marks: FirstGrade, Child, Female, Girl then done!

Note: FirstGrade gets marked since all the negative atoms in the clause (none) are marked

Observe:

- only letters in KB can be marked, so at most a linear number of iterations
- not goal-directed, so not always desirable
- a similar procedure with better data structures will run in *linear* time overall

First-order undecidability

Even with just Horn clauses, in the first-order case we still have the possibility of generating an infinite branch of resolvents.

KB:

$\mathrm{LessThan}(\mathrm{succ}(x),y) \Rightarrow \mathrm{LessThan}(x,y)$

Query:

$\mathrm{LessThan}(\mathrm{zero},\mathrm{zero})$

As with full Resolution, there is no way to detect when this will happen

$[\neg\mathrm{LessThan}(0,0)]$

$\downarrow$ $x/0,\ y/0$

$[\neg\mathrm{LessThan}(1,0)]$

$\downarrow$ $x/1,\ y/0$

$[\neg\mathrm{LessThan}(2,0)]$

$\downarrow$ $x/2,\ y/0$

...

There is no procedure that will test for the satisfiability of first-order Horn clauses

the question is undecidable

As with non-Horn clauses, the best that we can do is to give control of the deduction to the <u>user</u>

to some extent this is what is done in Prolog, but we will see more in “Procedural Control”