4.2 Scheduling to Minimize Lateness

SUBSECTION 4.2 OF KT's BOOK

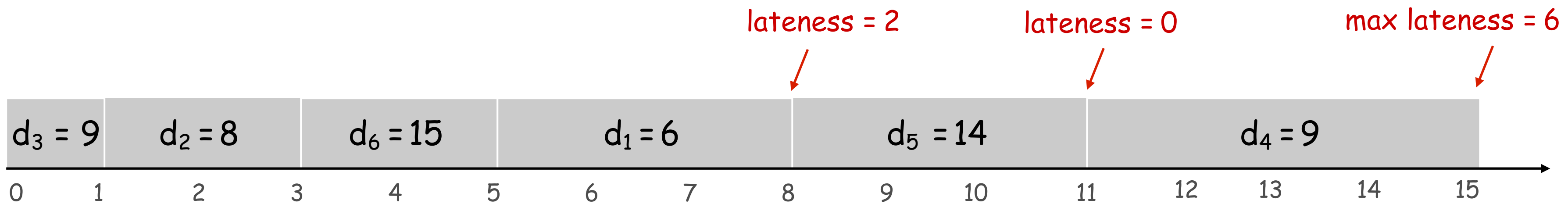
Scheduling to Minimizing Lateness

Minimizing lateness problem.

- **Single** resource processes **one** job at a time.
- Job j requires t_j units of processing time and is due at time d_j.
- Solution: If j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- Lateness: $l_j = \max\{0, f_j - d_j\}$.
- ***Goal:*** *schedule all jobs to minimize maximum lateness* $L = \max l_j$.
- **Note: input** elements are **in blue, solution** elements are **in red, cost** elements are in **violet**

Ex:

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [*Shortest processing time first*] Consider jobs in ascending order of processing time t_j.

- [*Earliest deadline first*] Consider jobs in ascending order of deadline d_j.

- [*Smallest slack*] Consider jobs in ascending order of slack $d_j - t_j$.

Minimizing Lateness: Greedy Algorithms

Greedy template. *Consider jobs in some order.*

- [**G1**: Shortest processing time first] Consider jobs in **ascending** order of processing time t_j.

	1	2
t_j	1	10
d_j	100	10

counterexample

G1 solution: Job 1; Job 2 --> Latency = 1

Optimal Solution: Job 2; Job 1 --> Latency = 0

Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [G2 Smallest slack] Consider jobs in **ascending** order of slack $d_j - t_j$.

G2 Solution: Job 2; Job 1. Latency = 10

Optimal: Job 1; Job 2. Latency = 1

	1	2
t_j	1	10
d_j	2	10

counterexample

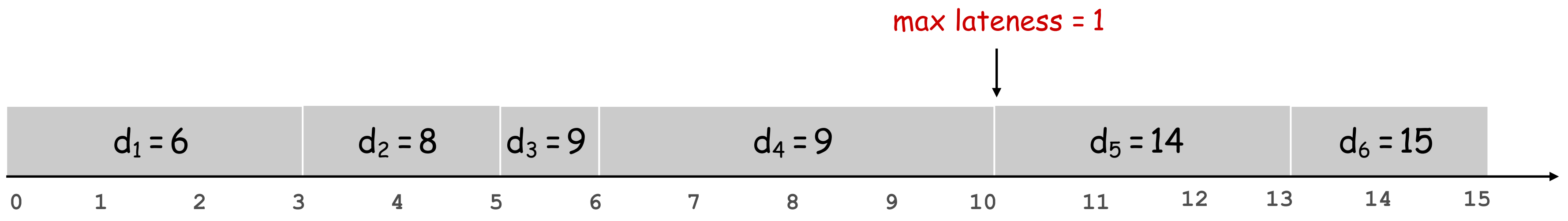
Minimizing Lateness: Greedy Algorithm

Greedy algorithm. Earliest deadline d first

Input: { (t_1,d_1), ..., (t_j,d_j),....(t_n,d_n) }

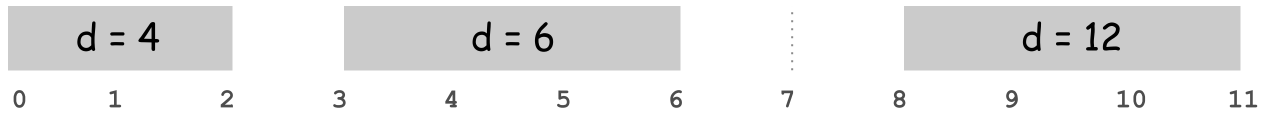
```
Sort n jobs by deadline so that $d_1 \le d_2 \le \dots \le d_n$

t ← 0
for j = 1 to n
   Assign job j to interval [t, t + $t_j$]
   $s_j$ ← t, $f_j$ ← t + $t_j$
   t ← t + $t_j$
output intervals [$s_j$, $f_j$]
```

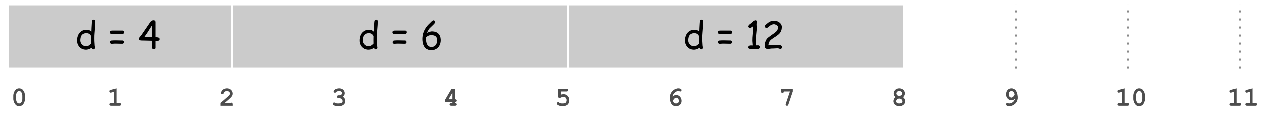
	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Minimizing Lateness: No Idle Time

Observation. There exists an **optimal** schedule with no idle time.

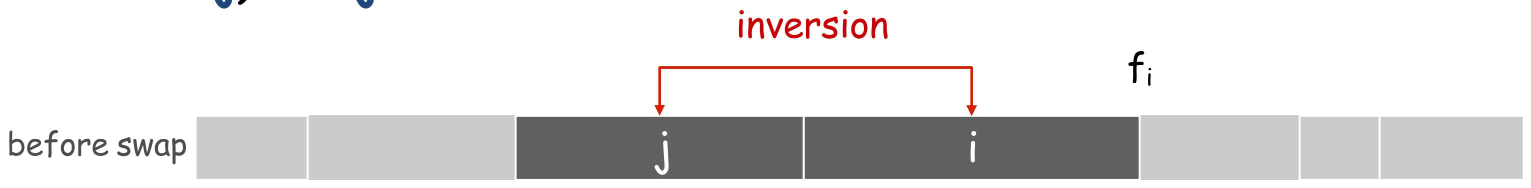




Observation. *The greedy schedule has no idle time.*

Minimizing Lateness: Inversions

Def. Given a **schedule** S, an **inversion** is a pair of jobs i and j such that: i< j (i.e. di <= dj) but j scheduled before i.



[as before, we assume jobs are numbered so that $d_1 \le d_2 \le \ldots \le d_n$]

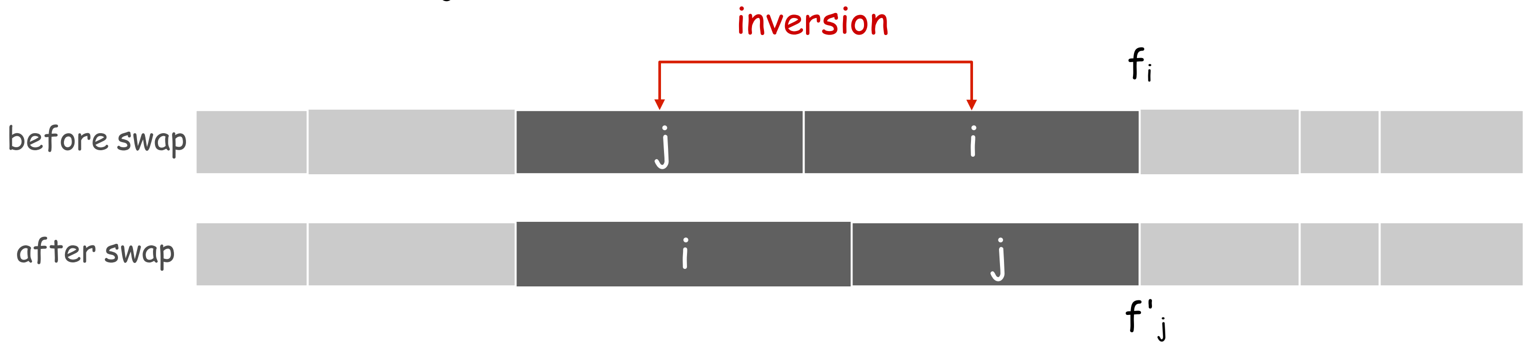
Observation. *Greedy schedule has no inversions.*

Observation. *If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled **consecutively**.*

a ≤ b ≤ c .... c' : c". ... <= f : f' :
If b > f' then for some consecutive c', c" it must holds c' > c"

Minimizing Lateness: Inversions

Def. Given a schedule S, an **inversion** is a pair of jobs **i** and j such that:
i < j (w.r.t. deadline d) but **j is** scheduled <u>before</u> **i**



LEMMA (Exchange Arg.). Swapping two consecutive, inverted jobs **reduces** the number of inversions by **one** and does not increase the max lateness (the sum is commutative!).

- **Pf.** Let L be the lateness before the swap, and let L ' be it afterwards.

- $l'_k = l_k$ for all $k \neq i, j$
- $l'_i \leq l_i$
- If job j is late: →

$l'_j = f'_j - d_j$	(definition)
$= f_i - d_j$	(j finishes at time f_i)
$\leq f_i - d_i$	($i < j$)
$= l'_i$	(definition)

Minimizing Lateness: Analysis of Greedy Algorithm

Theorem. Greedy schedule S is optimal.

Pf. Define **S*** to be an optimal schedule that has the **fewest**° number of **inversions**, and let's see what happens.

- Can assume **S*** has no **idle time**.
- If S* has **no** inversions, then S = S*.
- If S* has an inversion, let i-j be an **adjacent** inversion.
  - swapping i and j <u>does not increase</u> the maximum lateness and strictly <u>decreases</u> the **number of inversions**
  - this contradicts definition° of S* ▪

Greedy Analysis Strategies

Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

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EXCERCISE 1 – PAGE 183

EXERCISE I: Prove that the Greedy Algorithm based on the **earliest finish time** is **optimal.**

SOLUTION: Let $A = \{i_1, i_2, \dots i_k\}$ denote set of jobs selected by ***Greedy;***

Let $J = \{j_1, j_2, \dots J_m\}$ denote set of jobs in an optimal solution. We know .....

Lemma 1 (*Greedy Stays Ahead*). For any $r = 1, \dots, k$ it holds $f(i_r) <= f(j_r)$

- Now, suppose (by contradiction) that optimal solution is such that **$m >= k+1$**. So,

$$J = \{ j_1, j_2, \dots, j_k, j_{k+1} \dots, j_m \}$$

- Apply **Lemma 1** on intervals i_k and j_k: → $f(j_k) >= f(i_k)$. (*)

- From (*), we get that the Greedy would have inserted j_{k+1} too! Since it is <u>compatible</u> with i_k as well! Contradiction with the assumption $|A| = k$!

EXCERCISE 2 AT PAGE 185

-BUYING ITEMS OF INCREASING COSTS

DO AS HOMEWORK!

HINTS: DON'T LOOK AT THE BOOK, TRY GREEDY SOLUTIONS, and PROVE :

- INVERSIONS in the good greedy ORDERING imply CONTRADICTIONS
- How prove CONTRADICTIONS? exchange argument