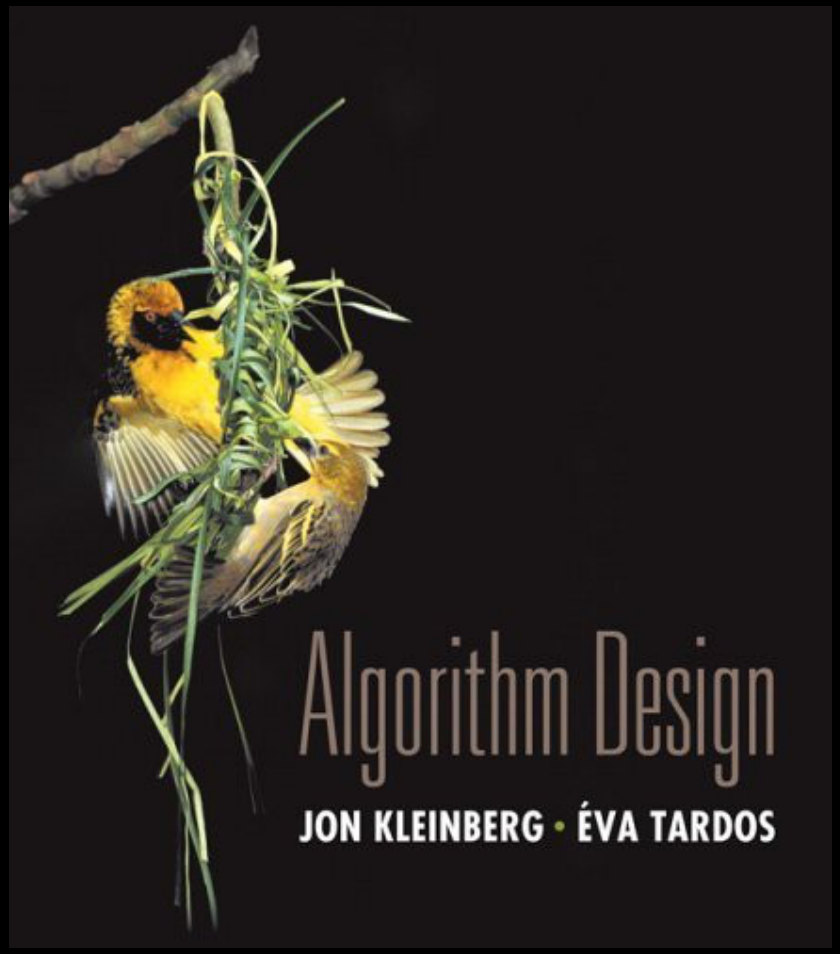


Chapter 8

NP and Computational Intractability

Slides by Kevin Wayne.

Algorithm Design Patterns and Anti-Patterns

Algorithm design patterns.	Ex.
. Greedy.	$O(n \log n)$ interval scheduling.
. Divide-and-conquer.	$O(n \log n)$ FFT.
. Dynamic programming.	$O(n^2)$ edit distance.
. Reductions.	
. Local search.	
. Randomization.	

Algorithm design anti-patterns.	
. NP-completeness.	$O(n^k)$ algorithm unlikely.
. PSPACE-completeness.	$O(n^k)$ certification algorithm unlikely.
. Undecidability.	No algorithm possible.

8.1 Polynomial-Time Reductions

Classify Problems According to Computational Requirements

Q. Which problems will we be able to solve in practice?

A working definition. [von Neumann 1953, Godel 1956, Cobham 1964, Edmonds 1965, Rabin 1966]
Those with polynomial-time algorithms.

Yes	Probably no
Shortest path	Longest path
Matching	3D-matching
Min cut	Max cut
2-SAT	3-SAT
Planar 4-color	Planar 3-color
Bipartite vertex cover	Vertex cover

Primality testing	Factoring

Classify Problems

Desiderata. Classify problems according to those that can be solved in polynomial-time and those that cannot.

Provably requires exponential-time.

- Given a **Turing machine**, does it halt in at most **k** steps?
- Given a board position in an **n-by-n** generalization of **chess**, can **black** guarantee a win?

Frustrating news. Huge number of fundamental problems have defied classification for decades.

This chapter. Show that these fundamental problems are "**computationally equivalent**" and appear to be different manifestations of one **really hard** problem.

Polynomial-Time Reduction

Desiderata'. Suppose we could solve Y in polynomial-time. What else could we solve in polynomial time?

don't confuse with reduces from

Reduction. Problem X polynomial reduces to problem Y if arbitrary instances of problem X can be **solved** using:

- Polynomial number of standard computational steps, **plus**
- Polynomial number of ***calls*** to ***oracle*** that solves problem Y.

computational model supplemented by special piece of hardware that solves instances of Y in a single step

Notation. $X \leq_P Y$.

Remarks.

- We pay for time to **write down** instances sent to black box $\Rightarrow$ instances of Y must be of ***polynomial size in the instance of*** X.
- **Note:** ***Cook*** reducibility.

in contrast to **Karp reductions**

Polynomial-Time Reduction

Purpose. Classify problems according to relative difficulty.

The use of poly-time **reduction**.

Design algorithms. If $X \leq_P Y$ **and** Y can be solved in polynomial-time, then X can also be solved in polynomial time.

Establish intractability. If $X \leq_P Y$ **and** X cannot be solved in polynomial time, then Y cannot be solved in polynomial time as well.

Establish equivalence. If $X \leq_P Y$ and $Y \leq_P X$, we use notation $X \equiv_P Y$.

Reduction By Simple Equivalence

Basic reduction strategies.

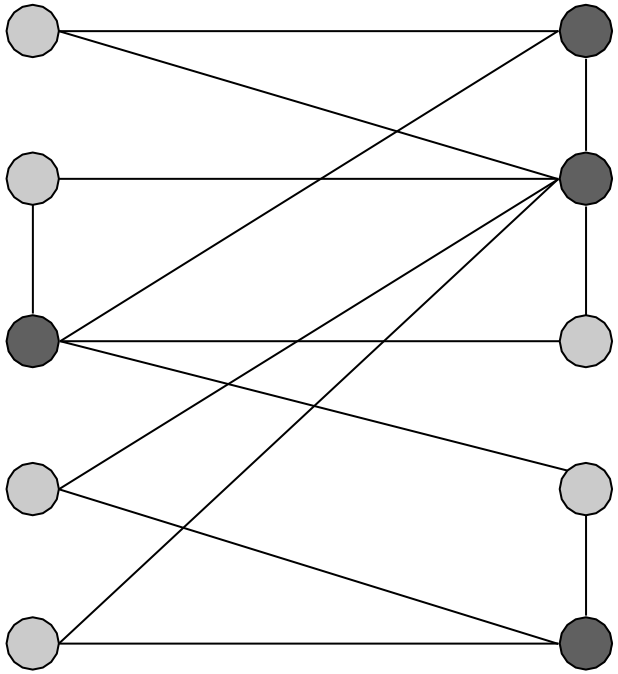
- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Independent Set

INDEPENDENT SET: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S?

Ex. Is there an **independent set** of size ≥ 6? **Yes.**

Ex. Is there an **independent set** of size ≥ 7? **No.**



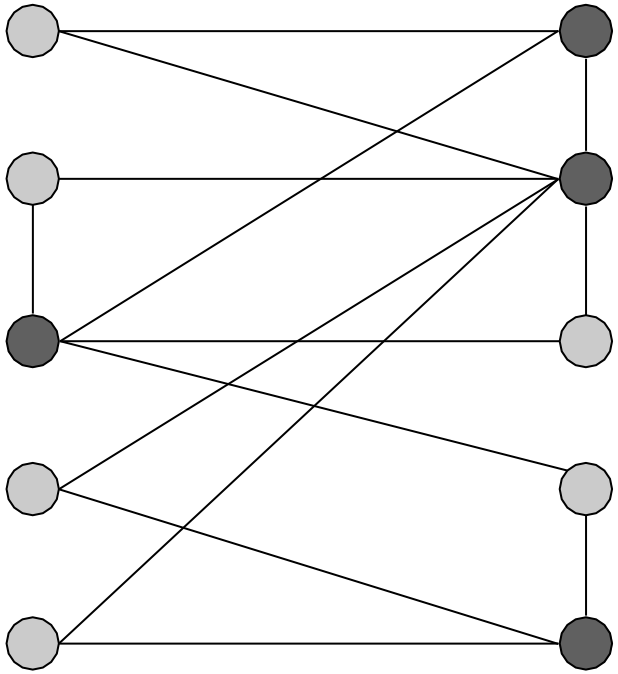
independent set

Vertex Cover

VERTEX COVER: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?

Ex. Is there a vertex cover of size ≤ 4? Yes.

Ex. Is there a vertex cover of size ≤ 3? No.



vertex cover

Hardness of Vertex Cover

VERTEX COVER: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?

Excercise: Greedy does not work!

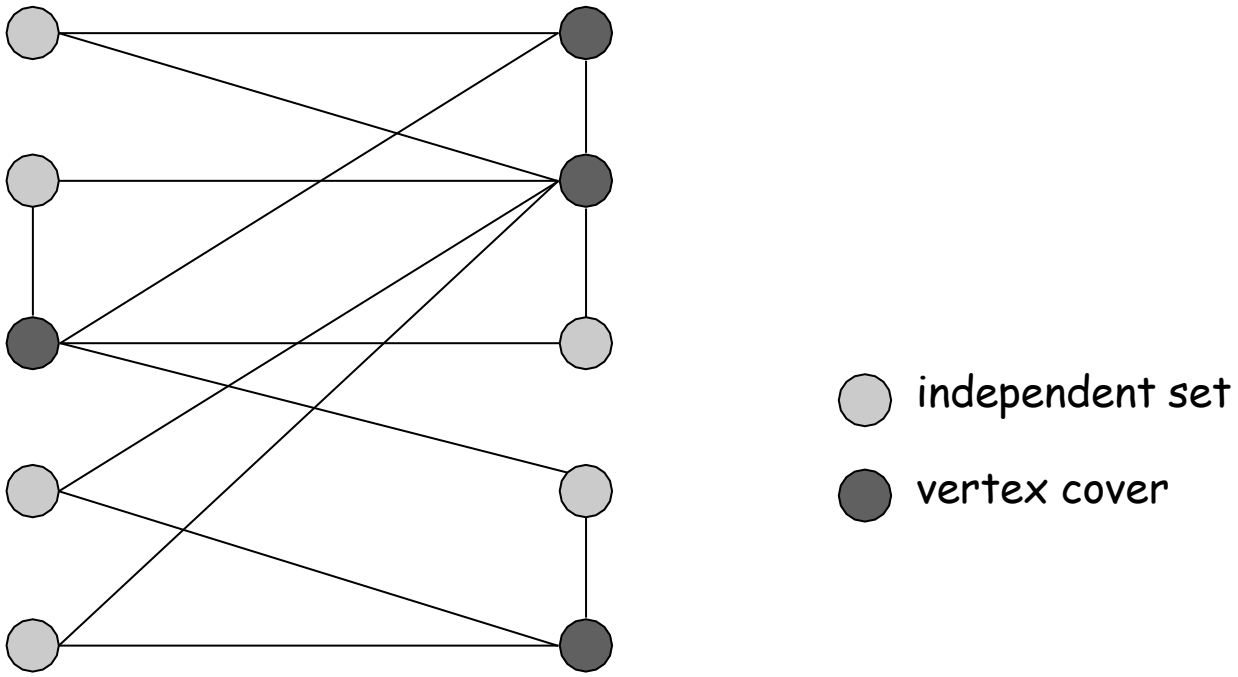
- 1) Select the node **v** with maximal degree (break ties arbitraly)
- 2) remove **v** and its edges
- 3) apply step 1 to the new graph until all edges are covered.

- Show the algorithm does not find (always) the optimum.

Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.

Pf. We show **S** is an independent set **iff** **V - S** is a vertex cover.



Vertex Cover and Independent Set

Claim 1. **S** is an independent set **iff** **V - S** is a vertex cover

Pf.

$\Rightarrow$

- Let **S** be any independent set.
- Consider an arbitrary edge **(u, v).**
- **S** independent $\Rightarrow u \notin S$ or $v \notin S \Rightarrow u \in V - S$ or $v \in V - S$.
- Thus, **V - S covers (u,v).**

$\Leftarrow$

- Let **V - S** be any vertex cover.
- Consider two nodes $u \in S$ and $v \in S$.
- Observe that $(u, v) \notin E$ since **V - S** is a vertex cover.
- Thus, no two nodes in **S** are joined by an edge $\Rightarrow$ **S** independent set. ▪

Vertex Cover and Independent Set

Thm. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.

Pr.

- $VC \leq_P IS$; Hypothesis: Poly-time Alg. A(**) for IS

- Given instance $X = \langle G(V,E) ; k>0 \rangle$ of VC;

- Transform it into instance $Y=\langle G(V,E) , k' = n-k \rangle$;

- Apply Algo A(Y) (thanks to Claim 1)

- $IS \leq_P VC$;

- Similar.

Reduction from Special Case to General Case

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Set Cover

SET COVER: Given a set of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, does there exist a collection of $\leq k$ of these sets whose **union** is equal to U?

Sample application.

- m available pieces of software $W_1, \ldots, W_m$.
- Set U of n capabilities that we would like our system to have.
- The ith piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all n capabilities using fewest pieces of software.

Ex:

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$k = 2$

$S_1 = \{3, 7\}$

$S_2 = \{3, 4, 5, 6\}$

$S_3 = \{1\}$

$S_4 = \{2, 4\}$

$S_5 = \{5\}$

$S_6 = \{1, 2, 6, 7\}$

Vertex Cover Reduces to Set Cover

Claim. VERTEX-COVER $\leq_P$ SET-COVER.

Pf. Given a VERTEX-COVER instance **G = (V, E), k**, we construct a set cover instance whose size equals the size of the vertex cover instance.

Construction.

- GOAL of reductions:

Programming with the language of the second problem

VC Language: *Q*. What means when we select a node **v** in **V**?

A. We cover **all** its edges

SC Language: 1 edge → 1 element of the Universe **U**

1 vertex **v** → 1 subset of U = edges covered by **v**

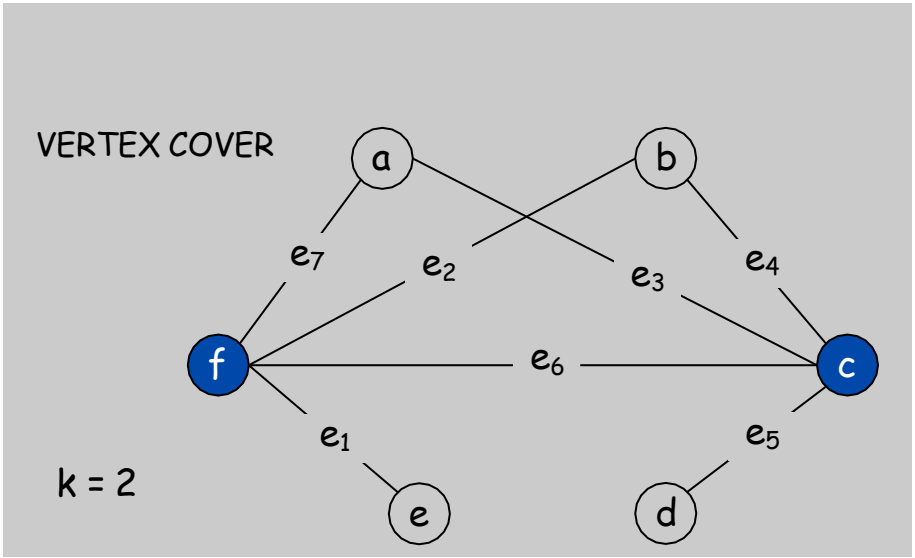
. Create SET-COVER instance:

- **k = k, U = E, for any v ∈ V: S_v = {e ∈ E : e incident to v}**

Vertex Cover Reduces to Set Cover

Claim. VERTEX-COVER $\leq_P$ SET-COVER.

- Create SET-COVER instance:
  - $k = k$, $U = E$, for any $v \in V$: $S_v = \{e \in E : e \text{ incident to } v\}$
- **Set Cover** of size $\leq k$ *iff* **Vertex Cover** of size $\leq k$. ▪



SET COVER

U = { 1, 2, 3, 4, 5, 6, 7 }
k = 2

$S_a = \{3, 7\}$ $S_b = \{2, 4\}$
$S_c = \{3, 4, 5, 6\}$ $S_d = \{5\}$
$S_e = \{1\}$ $S_f = \{1, 2, 6, 7\}$

Polynomial-Time Reduction

Basic strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

8.2 Reductions via "Gadgets"

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction via "gadgets."

Satisfiability

Literal: A Boolean variable or its negation. x_i or $\overline{x_i}$

Clause: A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form: A propositional formula Φ that is the conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

Def. The size $|\Phi|$ is the number of clauses.

SAT: Given CNF formula Φ, does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

each corresponds to a different variable

Ex: $(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$

Yes: x_1 = true, x_2 = true x_3 = false.

3 Satisfiability Reduces to Independent Set

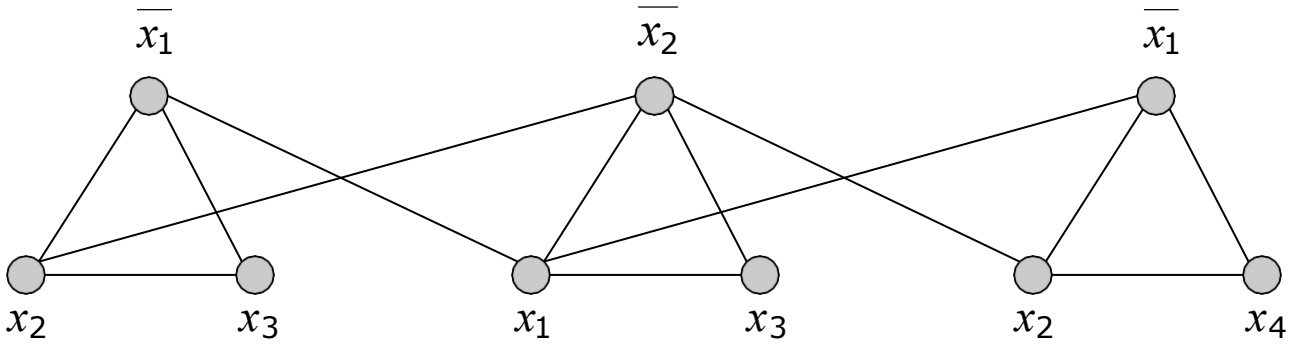
Claim. 3-SAT $\leq_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.

- G contains 3vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a **triangle**= GADGET
- Connect **positive** literal to each of its **negations**.
- set k = | Φ |

G



k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

3 Satisfiability Reduces to Independent Set

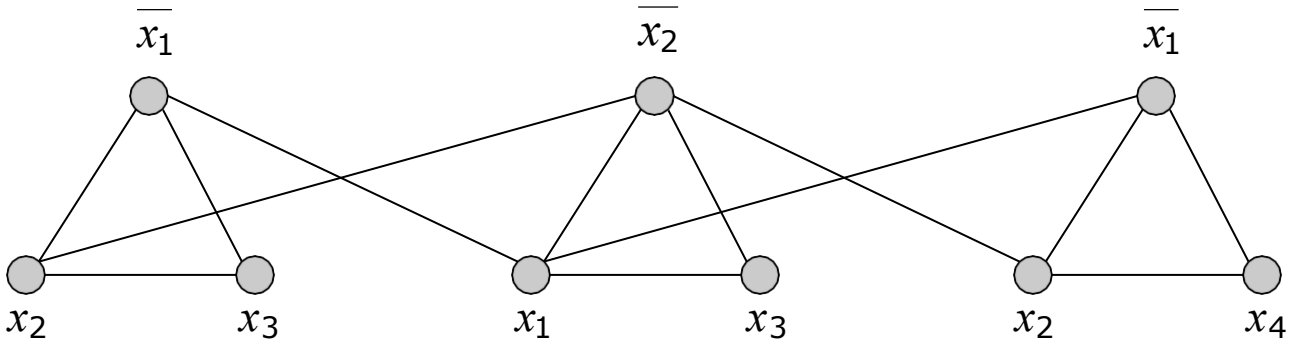
Claim. G contains independent set of size $k = |\Phi|$ iff Φ is satisfiable.

Pf. $\Rightarrow$ Let **S** be independent set of size k.

- **S** must contain exactly **one** vertex (= literal) in each **triangle**.
- Set these literals (= vertices) to **true**.
- No **adjacent** vertices in **S**
- **Truth** assignment is consistent and **all** clauses are satisfied.

Pf $\Leftarrow$ Given satisfying assignment, select one true literal from each triangle. No contradictions → This is an independent set of size k. ▪

G



k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

Review

Basic reduction strategies.

- Simple equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.
- Special case to general case: VERTEX-COVER $\leq_P$ SET-COVER.
- Encoding with gadgets: 3-SAT $\leq_P$ INDEPENDENT-SET.

Transitivity. If $X \leq_P Y$ and $Y \leq_P Z$, then $X \leq_P Z$.

Pf idea. Compose the two algorithms.

Ex: 3-SAT $\leq_P$ INDEPENDENT-SET $\leq_P$ VERTEX-COVER $\leq_P$ SET-COVER.

Self-Reducibility

Decision problem. Does there exist a vertex cover of size $\leq$ k?
Search problem. Find vertex cover of minimum cardinality.

Self-reducibility. Search problem $\leq_P$ decision version.

- Applies to all (NP-complete) problems in this chapter.
- Justifies our focus on decision problems.

Ex: to find min cardinality vertex cover.

- (Binary) search for cardinality k* of min vertex cover.
- Find a vertex v such that $G - \{ v \}$ has a vertex cover of size $\leq k^* - 1$.
  - any vertex in any min vertex cover will have this property
- Include v in the vertex cover.
- Recursively find a min vertex cover in $G - \{ v \}$.

delete v and all incident edges