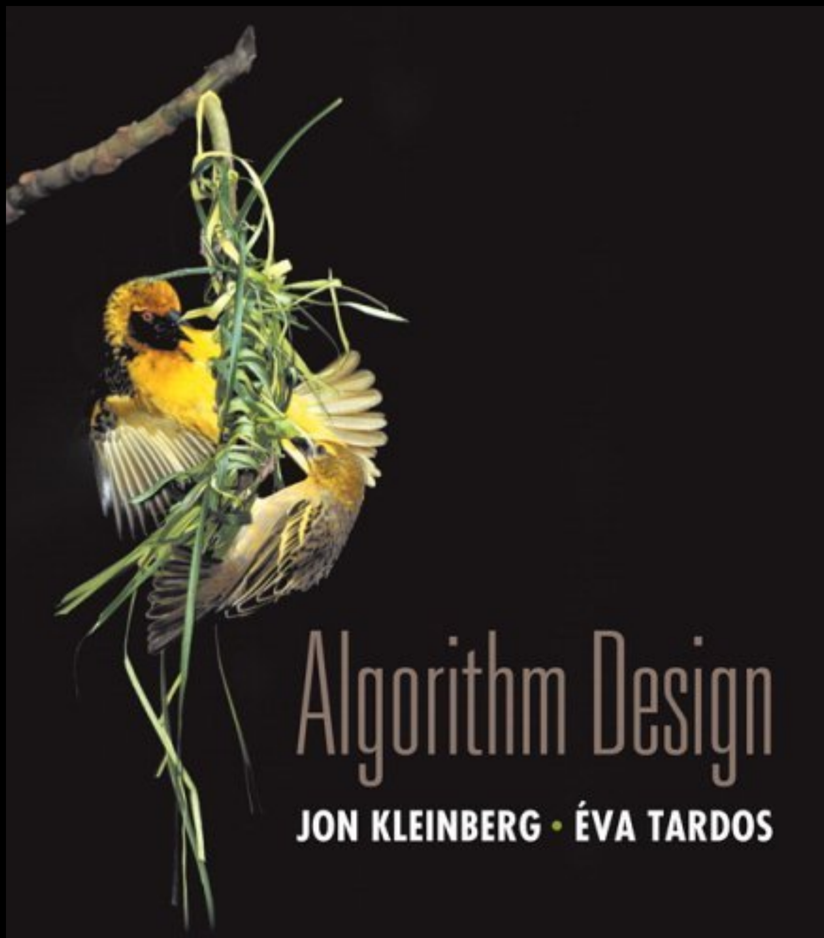


Chapter 8

NP and Computational Intractability

Slides by Kevin Wayne.

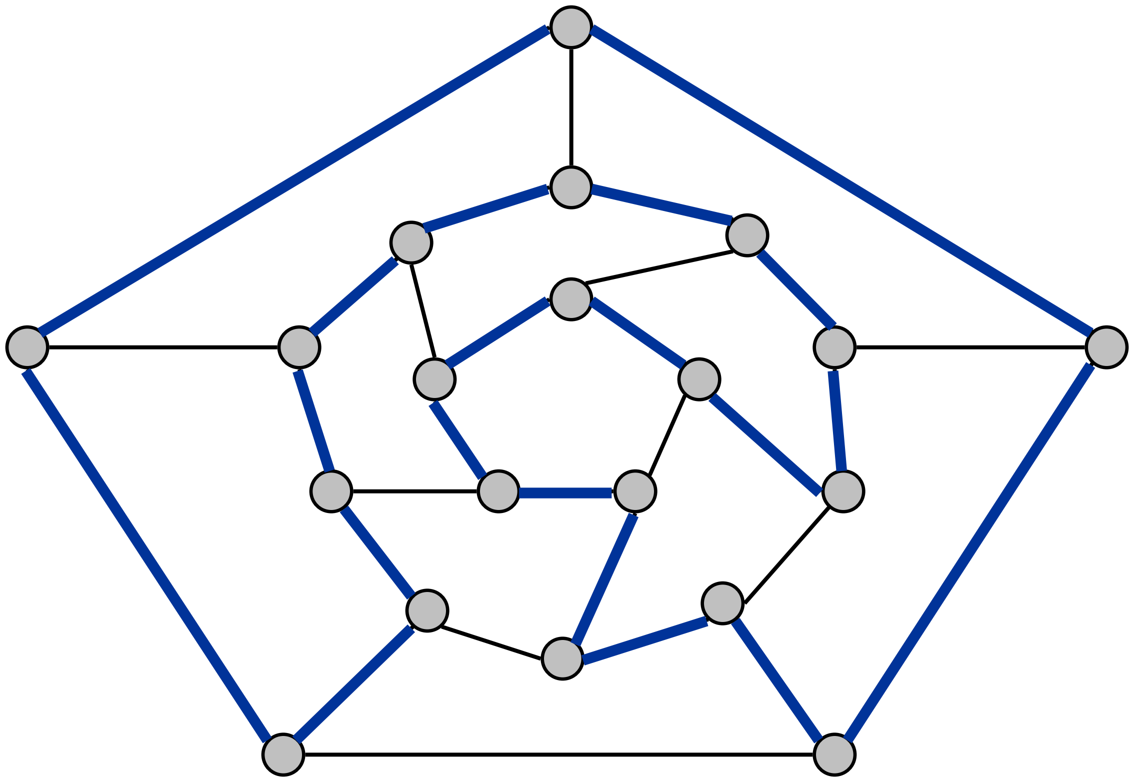
8.5 Sequencing Problems

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Hamiltonian Cycle

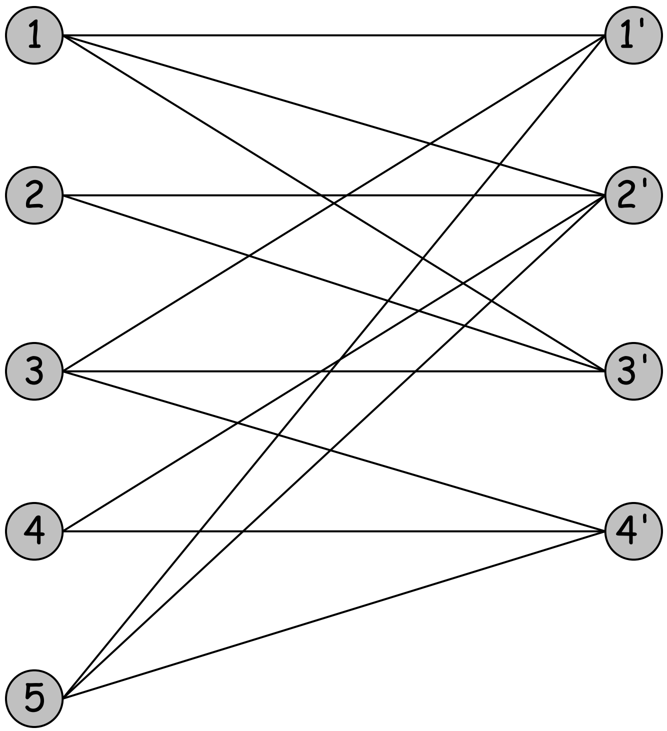
HAM-CYCLE: given an undirected graph **G = (V, E)**, does there exist a simple cycle Γ that contains every node in **V**.



YES Instance: vertices and faces of a dodecahedron.

Hamiltonian Cycle

HAM-CYCLE: given an undirected graph **G = (V, E)**, does there exist a simple cycle Γ that contains every node in **V**.



NO instance: bipartite graph with odd number of nodes.

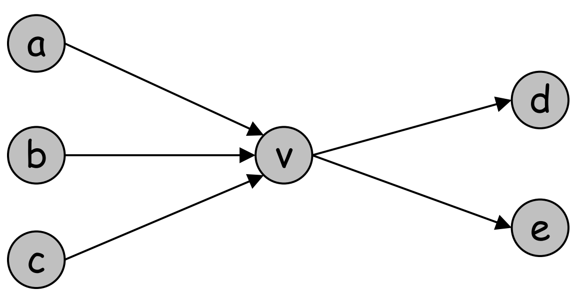
Directed Hamiltonian Cycle

DIR-HAM-CYCLE: given a **digraph** **G = (V, E)**, does there exists a simple directed cycle Γ that contains every node in **V**?

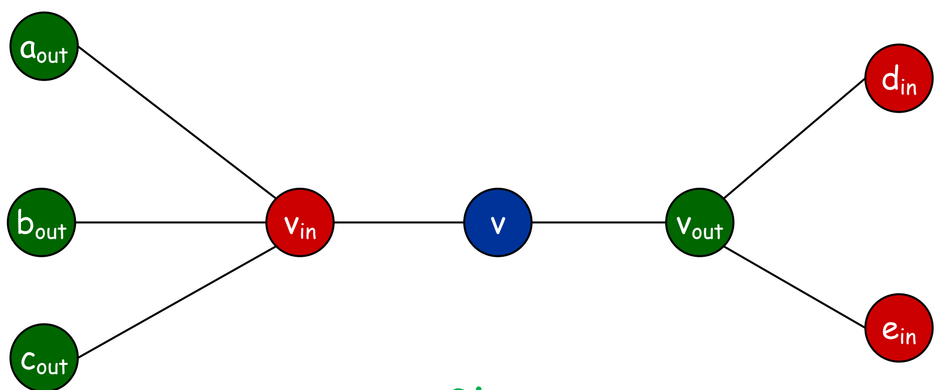
THM 1. DIR-HAM-CYCLE $\leq_P$ HAM-CYCLE.

Pf. Given a directed graph **G = (V, E)**, construct an undirected graph **G'** with 3n nodes.

GADGET for each node **v**



G



G'

Directed Hamiltonian Cycle

Proof of THM 1. *G* has a Hamiltonian cycle **iff** *G'* does.

Pf. $\Rightarrow$

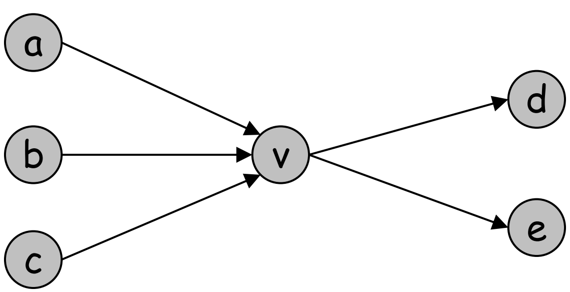
Suppose *G* has a directed Hamiltonian cycle Γ.
Then *G*' has an <u>undirected</u> Hamiltonian cycle (same order).

Pf. $\Leftarrow$

Suppose *G'* has an undirected Hamiltonian cycle Γ'.
Γ' must visit nodes in *G'* using one of following two orders:

…, B, G, R, B, G, R, B, G, R, B, … **clockwise** →
…, B, R, G, B, R, G, B, R, G, B, … **counterclockwise** ←

Blue nodes in Γ' make up <u>directed</u> Hamiltonian cycle Γ in *G*, or **reverse** of one. ▪



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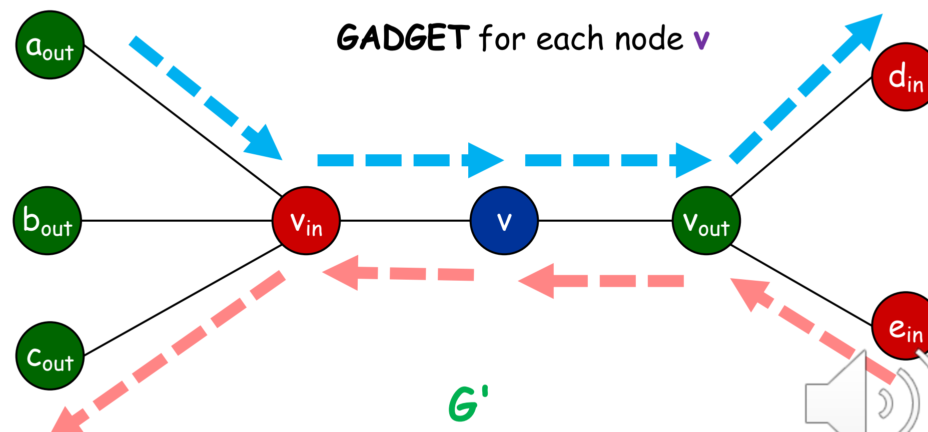


3-SAT Reduces to Directed Hamiltonian Cycle

THM 2. 3-SAT $\leq_P$ DIR-HAM-CYCLE.

Proof. Given an instance Φ of 3-SAT, we construct an instance G(V,E) of DIR-HAM-CYCLE such that:

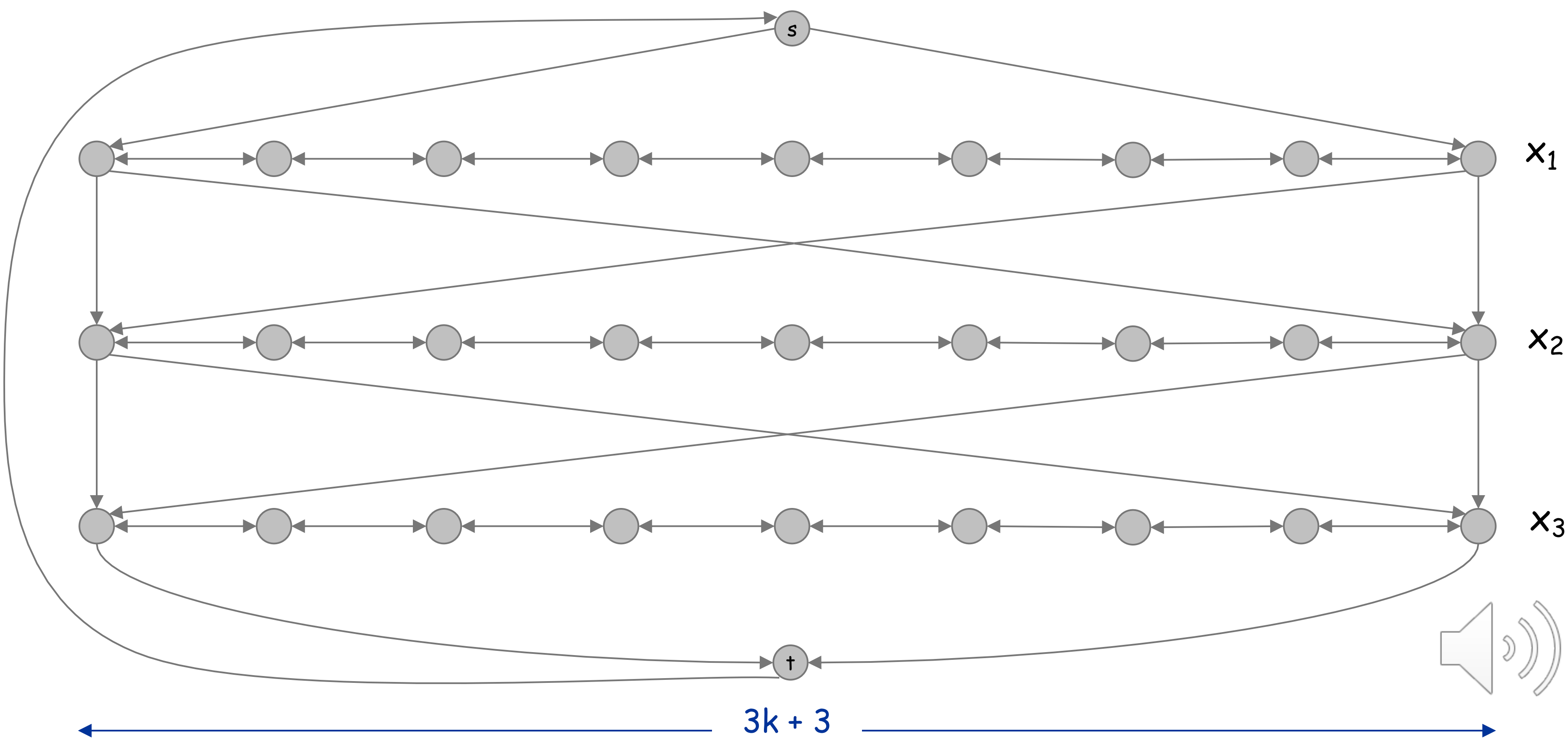
G has a **Hamiltonian cycle iff** Φ **is satisfiable.**

Construction. First, create graph that has 2^n Hamiltonian **cycles** which correspond in a natural way to 2^n possible truth assignments for Φ .

3-SAT Reduces to Directed Hamiltonian Cycle

Construction. Given **3-SAT** instance Φ with **n** variables x_i and **k** clauses.

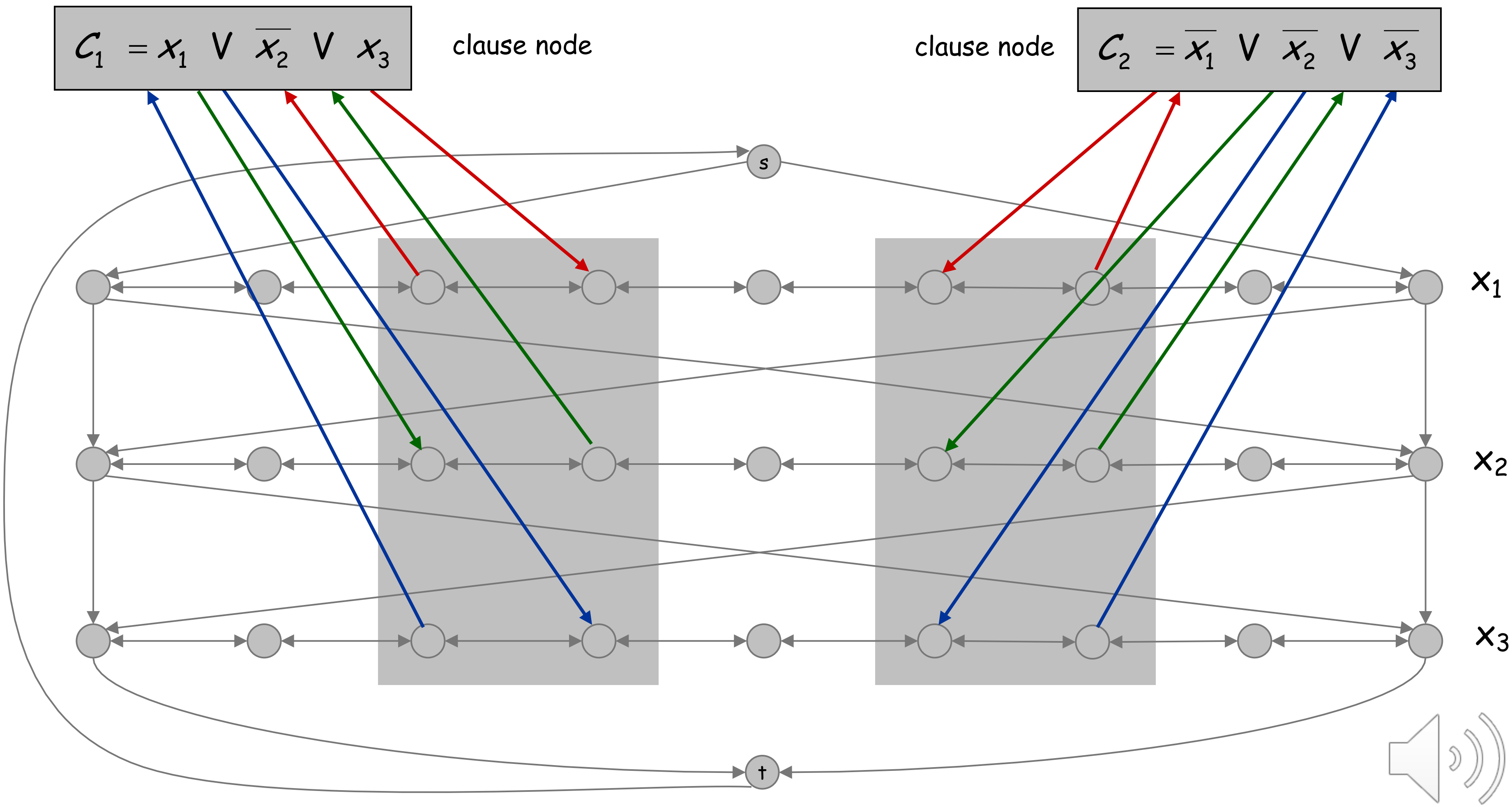
- Construct **G** to have 2^n Hamiltonian cycles.
- Intuition: traverse path **i** from left to right $\Leftrightarrow$ set variable $x_i = 1$.



3-SAT Reduces to Directed Hamiltonian Cycle

Construction. Given **3-SAT** instance Φ with **n** variables $\mathbf{x}_i$ and **k** clauses.

- For each **clause C_j**: add a ***gadget*** with 1 grey node and 6 edges.



THM. HAMILTONIAN-CYCLE ≤ HAMILTONIAN PATH

Proof (sketch). G(V,E) has a Hamilton Cycle iff **f(G)** has a Hamilton Path.

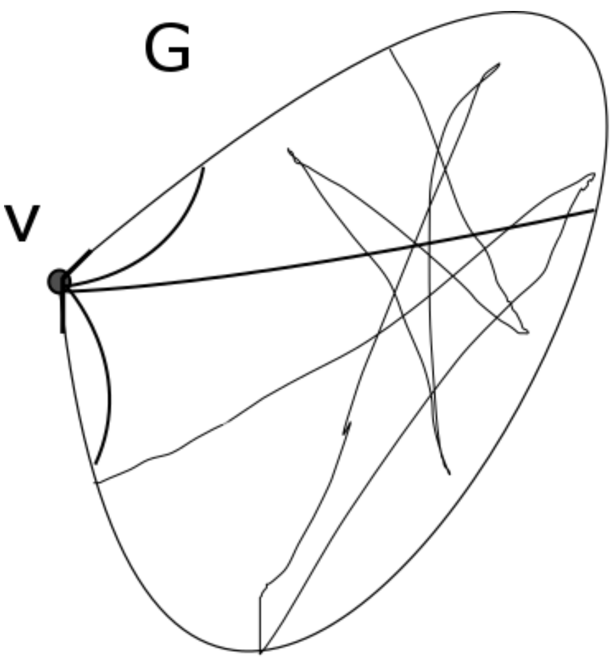
From **G=(V,E)** , construct **G'(V',E') = f(G)** as follows.

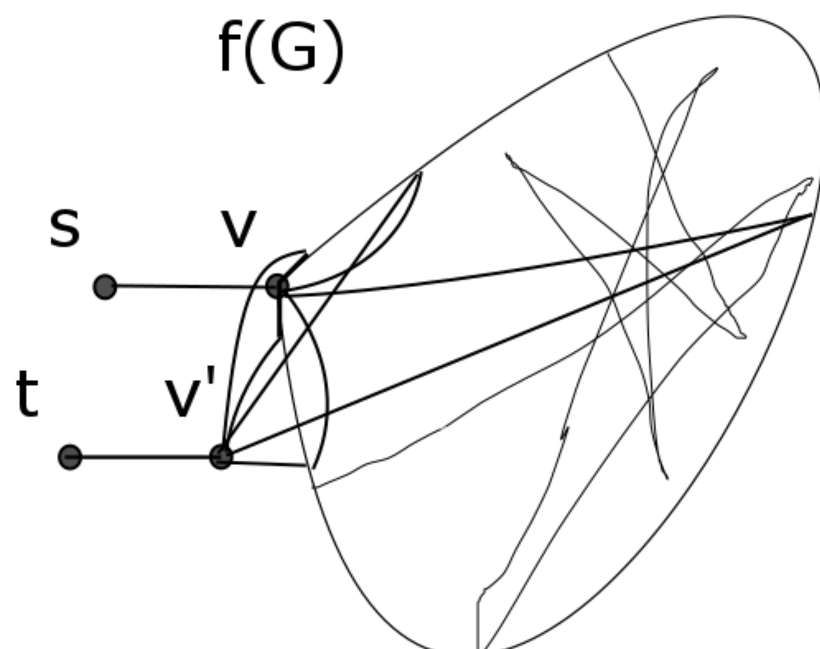
- Fix any **v∈V** and add 3 **new** nodes: **v',s,t∉V**.

'v' is a "copy" of **v,** and add a **source s** and a sink **t**, connected to **v**,**v'**, respectively. (See Figure)

Add **edges** $\{(v',w) | (v,w) \in E\} \cup \{(s,v),(v',t),(v,v')\}$.

If **G** has a Hamiltonian Cycle HC then it can be transformed into a Hamiltonian Path for **G'= f(G)** starting from **s** and ending to **t** and viceversa.





Longest Path

SHORTEST-PATH. Given a digraph **G = (V, E)**, does there exists a simple path of length **at most** **k** edges?

LONGEST-PATH. Given a digraph **G = (V, E),** does there exists a simple path of length **at least** **k** edges?

THM. **LONGEST-PATH is NP-Complete**

Proof.

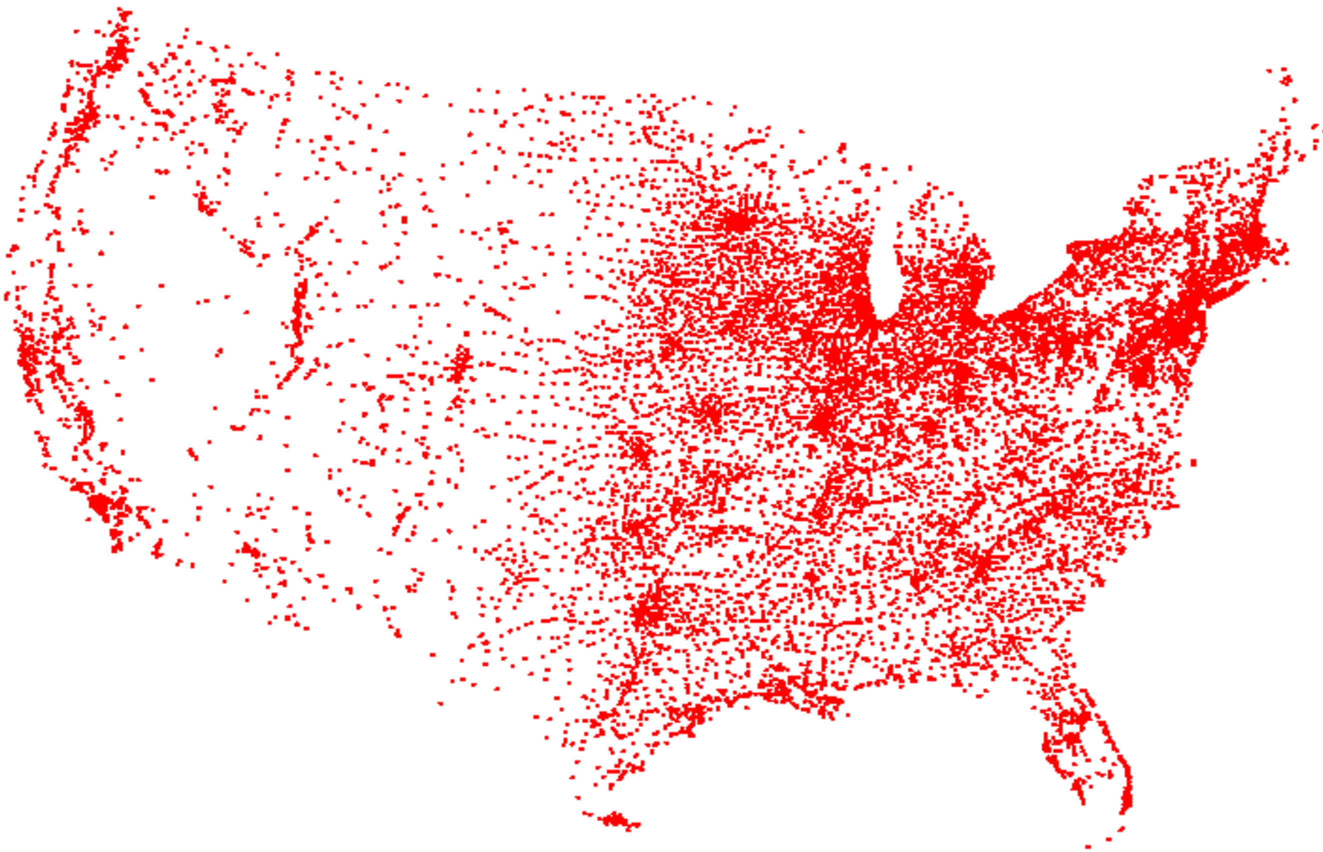
LEMMA: **HAM-PATH $\leq_p$ LONGEST-PATH:**

PROOF (of Lemma): Trivial, take: G(V,E) → < G(V,E), k=n-1>

HOMEWORK: Prove a direct reduction from **DIR-HAM-CYCLE**,
ignoring back-edge from **t** to **s**.

Traveling Salesperson Problem

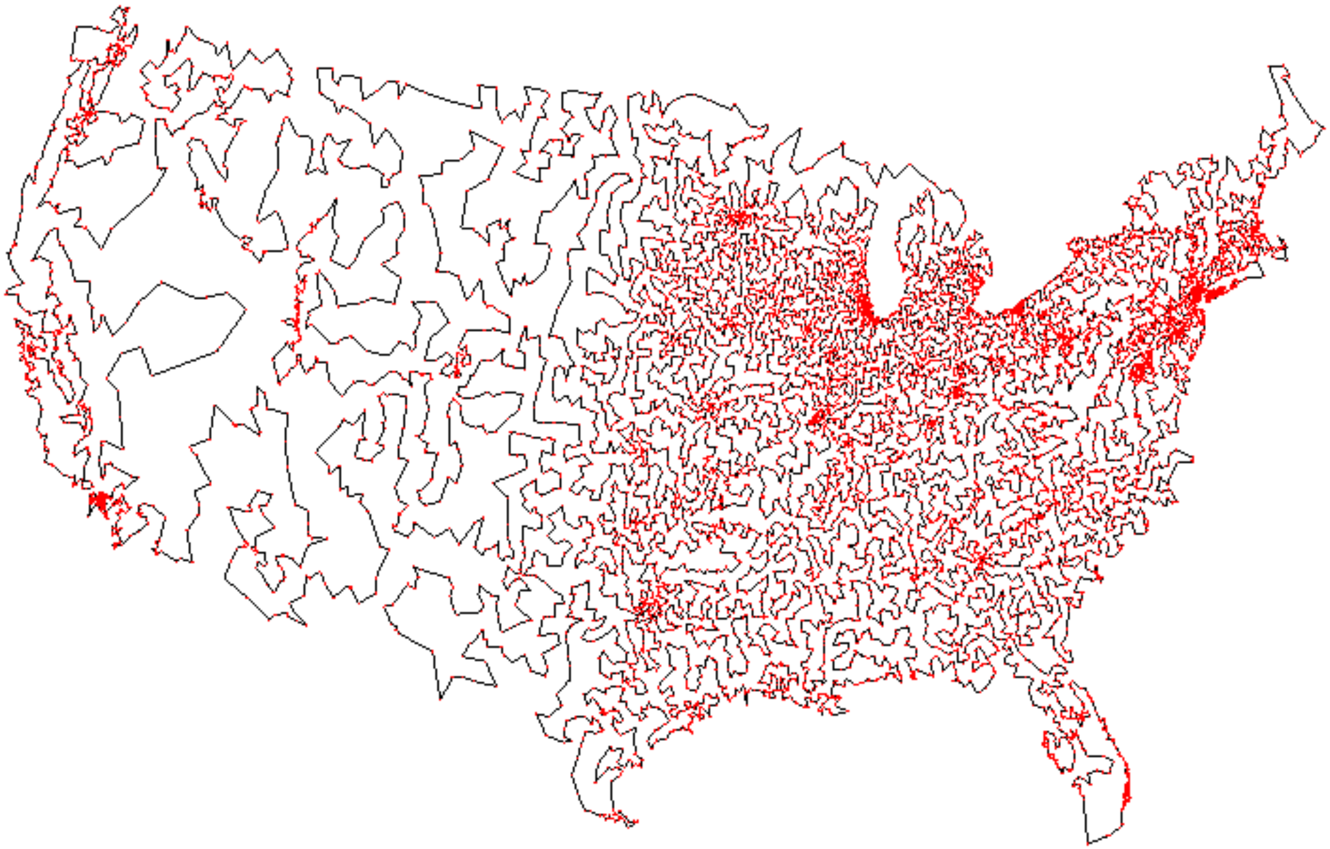
TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length $\leq$ D?



All 13,509 cities in US with a population of at least 500
Reference: http://www.tsp.gatech.edu

Traveling Salesperson Problem

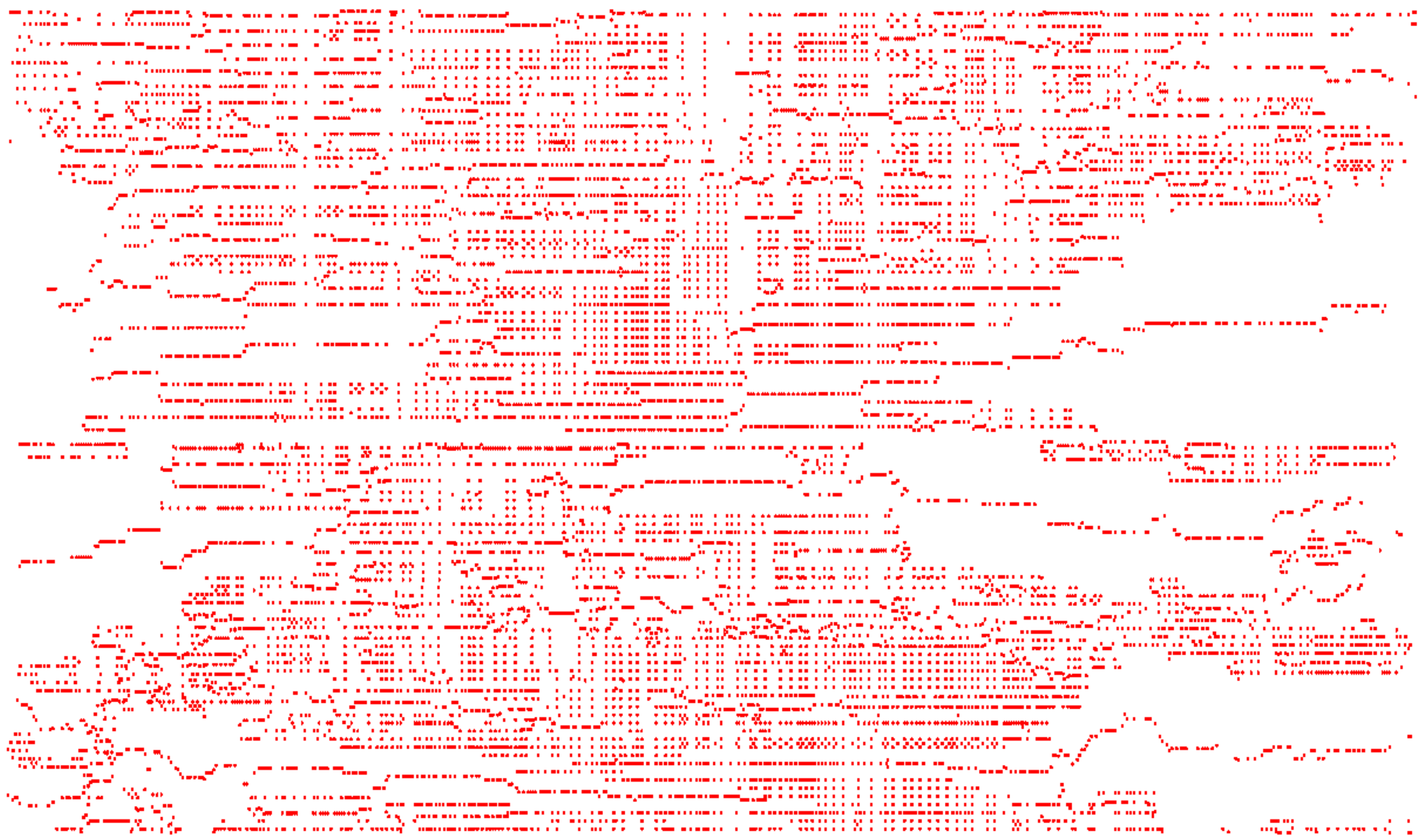
TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



Optimal TSP tour
Reference: http://www.tsp.gatech.edu

Traveling Salesperson Problem

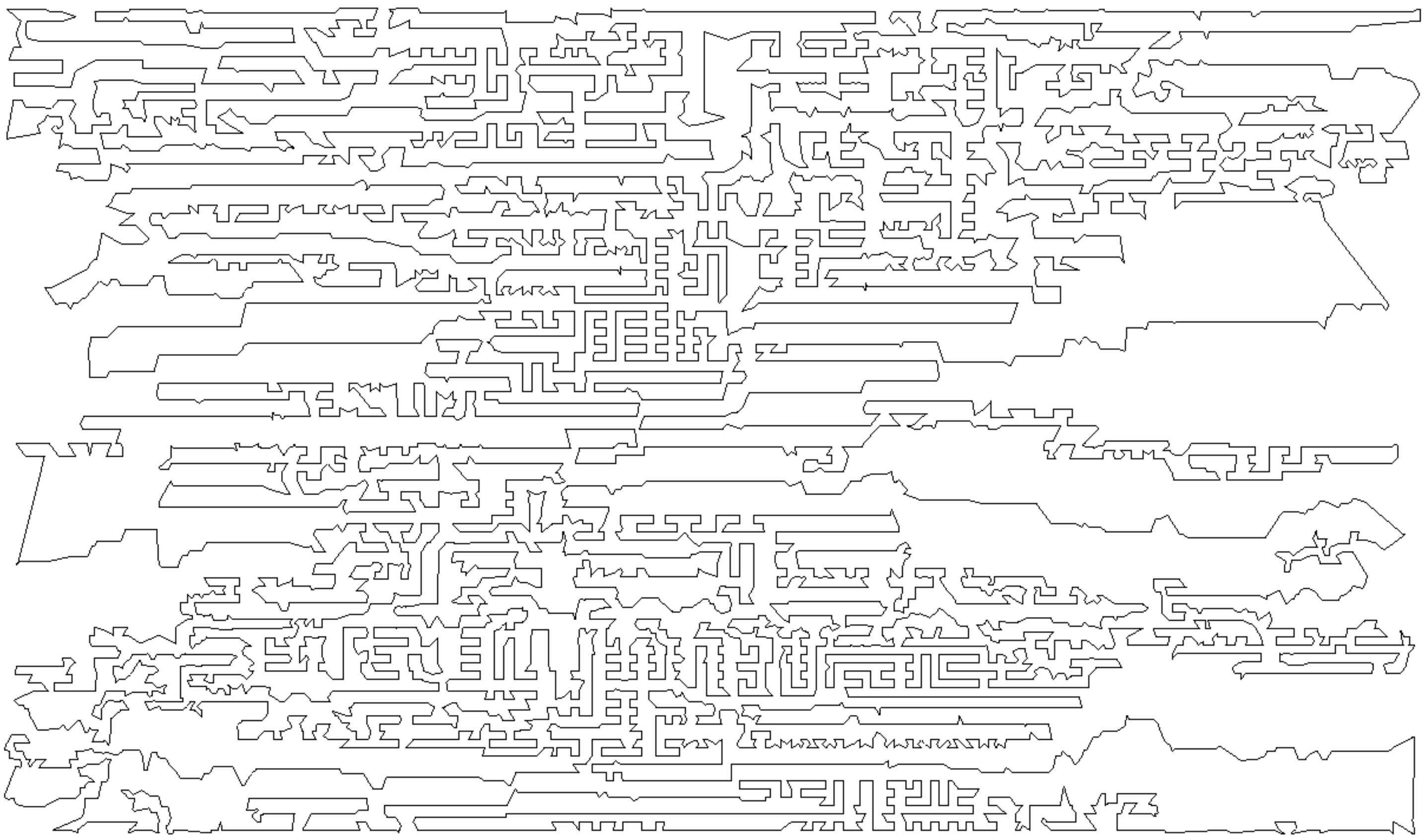
TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



11,849 holes to drill in a programmed logic array
Reference: http://www.tsp.gatech.edu

Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



Optimal TSP tour
Reference: http://www.tsp.gatech.edu

Traveling Salesperson Problem

TSP. Given a set of **n** cities and a pairwise distance function **d(u, v)**, is there a tour/cycle of length ≤ **D**?

HAM-CYCLE: given a graph **G = (V, E)**, does there exists a simple cycle that contains every node in V (i.e. a Hamiltonian Cycle)?

Claim. **HAM-CYCLE** $\leq_P$ **TSP**.
Prooff. Given instance G = (V, E) of HAM-CYCLE with |V| = n:

- create **n** cities with **distance** function

$$d(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ 2 & \text{if } (u, v) \notin E \end{cases}$$

- **TSP** instance has tour of length ≤ **n** iff **G** is Hamiltonian. ▪