Approximation Algorithms

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Review: Decision Problems vs Optimization Problems in NP

Given any **Opt** Problem **Min-P** $= (X, Y(x), m(x, Y(x)), MIN/MAX)$
we can always define the corresponding
Decision Problem k-P = $(X, Y(x), m(x, Y(x)), <= k\ (>= k))$

FACT (Definition).

the corresponding **Opt** problem **Min-P** is **NP-hard**
IFF
the decision problem **k-P** is **NP-Hard**

COR. IF P # NP and Min-P is NP-hard, THEN there is no poly-time deterministic algorithm for it.

Proof. By contradiction, if **Min-P** would have a poly-time algo **A(x)** then it can be used to decide **k-P , for any k on the same instance x !**

Approximation Ratio (Error)

Optimization Problem

Given an optimization problem $P = (X, Y(x), m(x, Y(x)), MIN/MAX)$,
we say A is an r -**approximation** algorithm for P if, **for any** instance $x \in X$,
the computation $A(x)$ returns a feasible solution y^A such that:

$$\frac{m(x, y^A)}{\mathbf{opt}(x)} \leq r \geq 1 \qquad \text{(in the case of } \mathbf{MIN}\text{)}$$

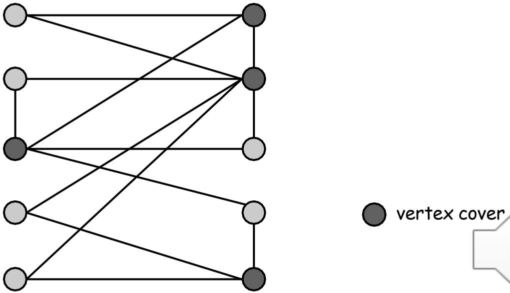
Min-Vertex Cover

k-VC: Given a graph $G = (V, E)$ and an integer k, is there a **k-size VC**, i.e., a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?

Min-VC: Given a graph $G = (V, E)$, find a VC S^* for G of **minimum size**

Ex. there **min-VC** for the graph below has size **4**.

4



Vertex Cover (VC)

A lower bound for optimal VC via Maximal Matchings

DEF. Given $G(V, E)$, a **Maximal Matching** $M \subseteq E$ *is a* ***maximal*** *subset of edges that share* ***no*** *vertex of* V.

FACT 1: Given any graph $G(V, E)$, consider any **Maximal Matching** $M \subseteq E$. Then, any **VC** for G must contain at least **1** vertex for every edge in M.

Proof.

immediate consequence of def.s of **Matching** and **VC**.

FACT 2 (Lower Bound for the Optimum): $\mathbf{opt}(G) \geq |M|$

Vertex Cover (VC)

An apx algorithm for Min VC

Matching Algorithm M−ALG

- Input: $G(V, E)$;
- Find (any) Maximal Matching M;
- Return $C = \{$all nodes touched by $M\}$;

Vertex Cover (VC)

An apx algorithm for Min VC

Matching Algorithm M−ALG

- Input: $G(V, E)$;
- Find (any) Maximal Matching M;
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THM 3. **M-ALG** is a **2-apx algorithm** for **Min-VC**.

Vertex Cover (VC)

Proof of THM 3.

FACT 4. The returned solution C is:
(1) always a **VC** for G AND **(2)** $|C| = 2|M|$.

Proof. (1) Immediate consequence of the fact that **M** is Maximal.
(2) is trivial. ⊡

Recall **FACT 2**: $\text{opt}(G) \geq |M|$ **(3)**

From (2) and (3), it follows that:

$$|C| \,/\, \text{opt}(G) \leq 2|M| \,/\, |M| = 2$$